Thoughts on Reason, Formalism, and Issues in Mathematics

Contents:

<u>**Thoughts on Reason, Formalism, and Issues in Mathematics**</u>

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Forward:
After reading a book by Morris Kline titled "Mathematics: The Loss of Certainty", I developed some thoughts respecting some of problems which that book propounded upon. For some time, I have intended to put these thoughts into presentable form. Finally having found the wherewithal to do so, I have produced the report below. However, due to other pressing issues requiring my attention, I have not been able fully to present my thoughts here, and some of the ideas that are presented could have been further elaborated upon and/or presented in a more effective form if time permitted. Nevertheless, it is my hope that the reader will find what has here been presented as fruitful in its consideration.

Introduction

There is perhaps nothing so widely utilized, believed and glorified, even while remaining essentially unknown and shrouded in mystery, as mathematics. Today, despite that similar definitions of the word "mathematics" are given[1] there is no universally accepted set of concepts or principles which can be said to comprise mathematics. There is no agreement as to what valid mathematics is, for there is no agreement as to what the principles of valid mathematical proof are. The history of mathematics is rife with controversies over questions as elementary as what should be considered a number. Today, there exist many different "schools" of mathematics, each with its own unique assortment of axioms and concepts which it accepts as valid.[2]

After becoming acquainted with the current dilemma of mathematics, and the historical development of mathematical thought which has led to it, I thought it worthwhile to put forward some points of view which might be of some helpfull value.

[1]From dictionary.com: "the systematic treatment of magnitude, relationships between figures and forms, and relations between quantities expressed symbolically."
[2] For a comprehensive and concise overview of the history of the great controversies in mathematical epistemology, see Morris Kline's excellent book "Mathematics: The Loss of Certainty".

I. Whence Mathematics?

There has been disagreement among philosophers for thousands of years as to how the human mind arrives at mathematical concepts. There are two principal schools of thought respecting this issue. One school of thought, exemplified by Aristotle, maintains that all concepts which the human mind possesses come from experience. For some philosophers, like Aristotle, who believe that the human mind is a "blank slate" devoid of characteristics at birth, this is a necessary conclusion. The other school of thought, exemplified by Plato, maintains that the concepts of mathematics are within the mind prior to any experience of them. This is coherent with the Platonic view that there are universal truths which the human mind has access to even prior to sensual experiences of them.

The approach of the Aristotelian school is repugnant to most reasonable people. Yet, as a we reason through what might be implied by the Platonic approach to the question, we are liable to arrive at equally repugnant or seemingly silly conclusions as well. For example, let us ask the following question: Is a circle a universal concept which exists in the mind prior to any sensual experience of something circular? One may be quick to respond that it must be so, for, after all, no one has ever seen something which was perfectly circular, and yet, everyone seems to have a very distinct concept of what a circle is- therefore, the concept could not have come from experience. Let us grant the truth of this for the sake of argument. This granted, what if we then proceeded to ask: Is a triangle a universal concept which exists in the mind prior to any sensual experience of something triangular? We may find that we are able to answer this question in the same way as we did for the circle- but, we are starting to get suspicious. If we proceeded to ask yet another question: Is a square a universal concept which exists in the mind prior to any sensual experience of something square-like? Again, we could answer in a way which mimicked our answers respecting the circle and triangle, but we would become even more uneasy. The reason that we would become more and more uneasy as this progression of questions -proceeded from triangle, to square, to pentagon, to hexagon, and on to every other conceivable figure of a given number of sides- continued is because we would realize that, by this reasoning, the mind is evidently packed with an infinite number of universal forms corresponding to any conceivable figure. We would need to conclude, for example, that the 49857294740595-agon, or the 89488833322-agon, or the googoleplex-agon were all universal forms in the mind, just as distinct and real as are the noble circle and triangle. What about other mathematical entities? If we adhere to the Platonic doctrine, are we not also obliged to admit that all of the numbers are also universal concepts in the mind? 1, the most noble of all, may be readily accepted as such. But what of 4, or 7, or 3848587? What about -1, or -9696? What about irrationals like the square root of 2? What about the special numbers like *i* or *e*? Indeed, if we be true Platonists, it seems we must admit that all these correspond to distinct universal forms which are packed into the mind of man and Creator, just waiting to be discovered.[3]

What is illustrated by the foregoing discussion is not the inherent inviability of a Platonic approach to the attempt to address the origin of mathematical concepts. Rather, it illustrates the problem associated with the question as to what constitutes a fundamental mathematical concept in the first place. Here we reach the problem addressed in the introduction, namely, that no one has yet been able to establish agreement as to what the fundamental mathematical concepts are. Despite the difficulties of addressing that question, the truth with which the Platonic conception of universality rings compels us to seek out answers consistent with that conception.

Discovering Principles

How are we to identify the true universal mathematical conceptions which can lay claim to coexistence with the human mind? The approach I have taken to this question is to pose an alternative question in its place: What are the fundamental characteristics of mind, or cognition, without which no cognition is conceivable? What are the principles of thought without which no thought can take place? What are the fundamental aspects of experience, without which no experience could occur? This will seem divergent from the popular conception of Platonic philosophy, for we are no longer seeking out a long list of universal forms which sit in the mind like objects in a shopping cart, but, rather, we are seeking out those fundamental principles of thought and experience from which all other concepts, including those of mathematics, derive their meaning and content.

For purposes of initial illustration, take the following set of symbols:

A A A A A A A A

If the question is asked: What is the one thing which all of these symbols correspond to? Any person person literate in the english language will reply: "These symbols are all markings for 'the letter A'." The Aristotelian may use such an inevitable response in support of his position that mathematical concepts are not inherent to the human mind, but are only adopted by convention. For instance, the Aristotelian could say: "You see, my Platonic friend, your

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[3] As William James put it: "When the first mathematical, logical, and natural uniformities, the first *laws*, were discovered, men were so carried away by the clearness, beauty and simplification that resulted, that they believed themselves to have deciphered authentically the eternal thoughts of the Almighty. His mind also thundered and reverberated in syllogisms. He also thought in conic sections, squares and roots and ratios, and geometrized like Euclid. He made Kepler's laws for the planets to follow; He made velocity increase proportionally to the time in falling bodies; He made the law of the sines for light to obey when refracted...He thought the archetypes of all things, and devised their variations; and when we rediscover any one of these his wondrous institutions, we seize His mind in its very literal intention."

argument for the preexistence of mathematical concepts in the mind which makes use of the fact that all people claim to know what a circle is even though they have never seen a perfect circle, but only many different but similar things which are all said to be circular- that argument, I say, is not sufficient. For we have here an instance of the same thing- all people claim to know what the letter 'A' is, despite the fact that they have never seen the perfect 'A', but only many different but similar symbols which are said to be 'A'. And surely you cannot believe the letter 'A' to be a universal form which exists eternally in the minds of God and man, for 'A' is nothing but an arbitrary squiggle on paper; a symbol blithely made up by people to represent a sound. Thus, who could possibly believe that 'A' is some universal form or eternal concept? Just as 'A' is an arbitrary figure which can be abstracted from many experiences of similar figures called 'A', so too is the circle merely an arbitrary figure abstracted from many different experiences of similar figures which are called "circle" or "circular". And it is the same for all the other figures of geometry! Therefore, just as 'A' cannot be said to be an eternally existing universal concept in the mind, so too, the circle, or any mathematical figure, cannot be said to be an eternally existing universal concept in the mind."

This is a somewhat compelling argument on the part of the Aristotelian. It is, in fact, true, in one sense, yet, also, false, in another sense. The demonstration of one aspect of the invalidity of this argument will serve to clarify what kinds of mathematical concepts can rightfully be considered universal in the Platonic sense.

The Principle of Equality

The proper response to this Aristotelian argument is, simply, to point out that the argument itself tacitly presupposes the existence of a universal mathematical concept in the mind. What concept is that? None other than the principle of equality. For if the observer of the various symbols for "A" did not possess, within their mind, the concept of equality, how could they ever judge one symbol of 'A', and another symbol of 'A' to be "similar"? For what is "similarity" but the convergence of different things upon conformity with the principle of equality? If no principle of equality existed in the mind prior to an observer's experience of things, then that observer would have no standard with which they could judge the relationship of those things to be "similar" or not. If no principle of relation existed in the mind prior to an observer's experience of things, then that observer would have no standard against which they could compare the experienced relationship of those things, and thus, no meaningful or intelligible judgement respecting the relation of those things could be made.[4] Thus, the ability of the mind to form arbitrary and

[4] No counter to this argument by the Aristotelian could be made which relied upon the assumption that such a notion of "equality" may have been put into the mind by the experience of two equal things, for no two things are perfectly equal. Not only that, but the Aristotelian would also need to admit that equal things of each specific kind thing had been experienced before any other experience of that type of thing could be judged to be similar. Not only would that be untenable due to the

artificial concepts, like "A", is dependent upon the ability of the mind to utilize its inherent principle of equality to recognize the similarity of multiple things which are all said, by convention, to be denoted (named) in the same way.[5]

This is not to say that a circle, or other geometrical figures, are merely artificial concepts like "A", which are best characterized by definition of the form: "That which is similar to the preceding things which have been given a certain name or label" Indeed, such artificial concepts/generalities are made possible by the true universal concepts, or principles, like equality. But they are conceptually vacuous as distinct objects for consideration. Geometrical figures, on the other hand, are more strictly *defined* by the universal concepts, insofar as the universal concepts are capable of applying to -that is, describing or defining the relations of- the things which constitute the elementary material of geometry- namely, the basic content of the visual domain: visual magnitudes (extensions). Artificial concepts are not precisely/absolutely defined by the self evident concepts such as equality.[6] Geometrical concepts are defined in absolute terms of the fundamental principles like equality, and are thus subject to logical analysis. The fact that these universal principles, like equality, are capable of being thought of in an absolute way, renders anything which is strictly defined by these principles subject to definite implications- that is, implications identifiable by the mind as necessarily proceeding from the way in which the absolute principles are combined to form the definition of the thing.

The Principle of Multiplicity

What other concepts, besides equality, can we identify as fundamental? If we look back to see what else was presupposed in the Aristotelian's argument about "A", we notice that the fundamental concept of equality was not the only concept which was presupposed. For the argument also presupposes the concept of multiplicity, or differentiation. For how could one admit that multiple different things were similar if one did not admit a multiplicity of things- a differentiation between things? How can *anything* be conceived which does not stand differentiated from something else? The very act of conceiving requires a conceiver and the

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fact that people can identify two things that they have never seen before as similar, but it would require that people had experienced an infinite number of types of equal things.

[5] This is, of course, not to say that creating artificial concept is a bad thing in all cases. For the establishment of artificial systems of convention (languages) greatly increases the efficiency of human communication.

[6] For a thing can be said to be distinguishable as one of a certain *type* by virtue of the fact that it is *similar* to other things bearing that label. Thus, there is "wiggle room" as to what is passable as a thing of a type. On the other hand, there is no room for discrepancy when we decide whether a thing is a circle or not. For a circle is *defined* as that distinct, finite (enclosed) geometrical area whose entire boundary is equidistant to one point. Anything which is not *perfectly* in accordance with this definition is not a circle (even though such could be said, sometimes, to be of the *type* named *circular*).

conceived- a dual multiplicity. Thus, the principle of differentiation, or multiplicity, is a fundamental principle of cognition- a necessary aspect of conscious thought.[7]

The Necessity of Gross Phenomenal Qualities

As a brief interruption to the discussion of the fundamental principles of cognition, I must make the following pont: No universal principles can be thought of as the prerequisite to the experience of any*thing* unless some*thing* is presupposed which the universal principles can apply to. For example, there is no meaning to the statement "The colors are equal." if one does not have a meaningful idea of what color is. Similarly, the statement: "The sounds are equal." has no meaning if one does not have a meaningful idea of what sound is. And so, to say "The areas are equal", which is a statement of geometry, has no meaning if one has no meaningful conception of geometrical areas. Thus, the universal principles/concepts in the mind require predication within some gross phenomena if they are to be consciously intelligible or meaningful. Sound, color, qualities of touch or kinesthetic experience, smells etc. are all gross phenomenal qualities.

This considered, we may be inclined to ask whether the phenomenal qualities of perception like sound, taste, or color should be included in the list of principles which are fundamental to experience and consciousness. For it seems impossible to imagine any consciousness or experience without them- or *some* kind of phenomenal experience. The answer is that, indeed, gross phenomena is a fundamental aspect of experience and consciousness. But, the various kinds of phenomena of experience are not necessary to consciousness in the same way as the fundamental principles are. For though *some* phenomenon is necessary, no *particular* phenomenon is. Consciousness is conceivable without color, and, indeed, people have been known who were blind from birth. Consciousness is conceivable without sounds, and indeed people have been known who born deaf. On the other hand, no experience or consciousness is conceivable without any one of the fundamental principles. The fundamental principles of cognition are dependent upon gross phenomena for their manifestation, and the experience of gross phenomena is, in turn, dependent upon the fundamental principles. But the dependence is not symmetric. The fundamental principles are the universal basis for the experience of all possible gross phenomena, but each gross phenomena is not universally requisite to the manifestation of the fundamental principles. For example, we can recognize the manifestation of equality by perceiving equal colors. But that is not the only way equality can be recognized- we may also perceive equal sounds, or tastes, or sensations of touch. The experience of the quality of taste is only possible with the universal principles, but experiencing a manifestation of the universal principles does not require the phenomenal quality of taste. No *particular* phenomenal quality is requisite to cognition, but *all* of the fundamental principles are. But, as said before,

[7] The principle of multiplicity is referred to by Nicholas of Cusa as "otherness" in his great work "De Conjecturis" (On Surmises). In this work, Cusa shows how it is that all conceivable things are contracted in otherness.

there must be *some* phenomenal quality which allows for the contingent manifestation of the principles of cognition.[8]

The Principle of Oneness

We also notice that the universal principle of oneness, or unity, was presupposed in the argument. For how could any *thing* be compared to any other *thing* if no concept of oneness existed in the mind which enabled it to conceptualize any one individual thing? Anything that we can conceive of can only be conceived of as *one* thing. All things are one to the extent that they are conceivable, and all things are conceivable to the extent they are one. If something is not one, then it is not conceivable as one thing, and therefore cannot be conceived by the mind. The mind only experiences what it distinguishes, and so, if no principle of oneness existed in the mind, it could not distinguish any one thing, and no experience would take place. Thus, the principle of oneness is a fundamental principle of cognition- a necessary aspect of conscious thought and experience.

Interrelation of the Fundamental Principles

We will quickly note here that there is an interdependence between the fundamental principles of cognition. We see this with oneness and multiplicity: no one thing can be conceived of unless it is distinguished from something else, and yet, no thing could be said to be different from another thing, unless the things were first conceivable as individual (one) things. Further, no one thing could ever be conceived of as one thing without the multiple characteristics of beginning, middle, and end.

This interdependence should not be surprising since, after all, no fundamental principle of cognition could be said to be such if there were an instance of cognition, or conscious experience, in which one of the principles was not active in conjunction with the rest.

So far, we have identified three fundamental concepts which were presupposed by the argument put forward by the Aristotelian and demonstrated to be fundamental aspects of cognition itself: Oneness, Differentiation, and Equality.

The Principle of Comparison

If anything is to be distinguished from anything else, the things distinguished from each other must be compared. And, conversely, any things to be compared to each other must be

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[8] In his work *De Conjecturis,* Nicholas of Cusa elaborates upon this ordering of the necessary gross/phenomenal and fundamental/universal aspects of cognition as an aspect of the ordering necessary to the reconciliation of the seemingly contradictory notions of differentiated experience, and God.

distinguished from each other. Comparison, then, is a fundamental aspect of experience without which no experience or cognition were conceivable.

--*Comparison and Equality*

Upon reconsidering the concept of equality, we might be inclined to conclude that it is not truly fundamental, since equality seems to be only a specific case of the more general concept of *relation*, or comparison. For different things can have a relationship which is equal, or they could have a relationship which is unequal. We may compare two circles which are unequal in size such that we find one to be greater relative to the other, and the other lesser relative to the former. We might conclude, then, that the concept of greater and the concept of lesser must be added to the list of universal principles of cognition along with the concept of equality. However, this may not be the case. For these two notions, greater, and lesser, are not independent concepts- the existence of each is contracted to the other in an inescapable duality which arises as when there is a departure from absolute equality. For if two things are said to be unequal, then it is necessarily admitted that one is greater and the other is lesser in some respect. We might say that it is only to the *extent* that the two things diverge from equality that they can be said to be greater or lesser than each other. But here, again, we presuppose the notion of relation- for we said "the *extent* that the two…" The idea of a *greater* or *lesser* convergence on equality cannot be the basis of the concept of greater and lesser, for that would be circular.

How can we understand, then, the seemingly evident notions of greater and lesser extents of inequality? We may relate one thing and another thing, and find that the standard of equality is not met. But, in order for us to say that this divergence from the standard of equality is greater or lesser in extent, we would need to compare the divergence of the current relation from equality to the divergence of another relation from equality, and find there, again, a divergence from equality. If a divergence of equality were found in this comparison of the first relations,[9] then one of those relations could be said to be unequal to the greater extent, while the other could be said to be unequal to the lesser extent (relative to each other).

Take the following example: Two circles, A and B are compared in their size. Their relation, or comparison, of size, called RAB, is found to be unequal. Then, two other circles, C and D, are compared in size. Their relation of size, called RCD, is found to be unequal. Then, the relations themselves are compared (related). RAB is compared to RCD and they are found to be unequal. RAB is found to be more in conformity to the standard of equality than RCD. Thus, we would say that C and D are *unequal to a greater extent* than A and B, or, translated into other words, the difference in size between C and D is greater than the difference in size between A and B .

[9] One might say "this comparison of the comparisons" or "this relation of the relations".

Thus, the principle of relation, or comparison, must be included in the list of universal principles inherent to the operation of the mind and necessary for all experience, and that which might be called cognition, which is the activity of the mind.[10]

--*Comparison and Magnitude*
The function of comparison, or relation, presupposes the concept of comparability or relatability of things. Because the comparability of different things depends upon something which is common to them both -that is, something which can be said to be equal between them- there are different kinds of comparability, for their are different ways in which things are the same. For examples: Different lines might be equal in that they are both lines, but they may be different in their length or curvature. Different sounds might be equal in that they are sounds, but they may be different in their pitch and volume. Different sensations of touch may be equal in that they are sensations of touch, but they may be different in their softness or roughness. The three examples just given are cases of comparability in which the different things to be compared are equal in some respect, and, also, that the aspect of them which is unequal could, in principle, be conceived of as being made equal. For example, two lines are equal in that they are both lines, but could differ in their length or curvature; yet, the lengths and curvatures of these lines are conceivable, in principle, as capable of being made equal though modification. That is, two lines may be unequal in length or curvature, but they *could* be equal in length or curvature. Two sounds are equal in that they are sounds, but could differ in their pitch or volume; yet, the pitches or volumes of these sounds is conceivable, in principle, as capable of being made equal thorough modification. That is, two sounds may be unequal in pitch or volume, but they *could* be equal in pitch or volume. Two sensations of touch are equal in that they are sensations of touch, but could differ in their roughness or softness; yet, the roughness or softness of these sensations of touch is conceivable, in principle, as being made equal through modification. That is, two sensations of touch may be unequal in roughness or softness, but they *could* be equal in roughness or softness. Thus, one way in which things can differ from each other is to be equal in some respect, and unequal in another respect which is capable of being made equal. We call these qualitative and quantitative respectively. Thus, when things of the same quality are compared, they are compared quantitatively. The concept of magnitude is the concept of a thing which is subject to quantitative comparison, that is, something which is capable of being conceived of as greater or lesser.

On the other hand some things may be equal in one thing and unequal in other things the which are not capable of being made equal. For example, a color and a sound are equal in that they are

[10] Not all instances in which the mind makes comparisons, or, judgements about the relation of things, is there necessarily conscious awareness that this is happening. Indeed, the things which are related may not even be in the conscious experience of the mind which compares the things at the same instance. Wolfgang Kohler called such relations "transphenomenal contexts".

both phenomena, but there are no other things about color or sound which are capable of being conceived of as equal. The comparison of things on the basis of this kind of difference is called comparisons of quality. Since comparisons of this kind cannot take place in the way in which quantitative comparisons of things are made, the way in which things of differing quality are conceived of as related is through the development of scientific hypotheses which satisfy the mind's craving for conceptual coherence in its consideration of the various aspects of its experience.

Magnitude can be demonstrated to be a necessary aspect of our experience, a necessary aspect of cognition. That is, magnitude is the most basic manifestation of the fundamental principles of cognition within a gross/phenomenal subject. Not as a fundamental principle, but as the necessary way in which the gross qualities necessary to experience manifest themselves to the mind (are experienced) on the basis of the fundamental principles.

Here is the argument: As we pointed out earlier, in order for anything to be experienced or conceived of it must be distinguished from something else. For if no distinguishability of things were possible, then nothing could be distinguished, and thus no experience of any thing could take place. Thus, each thing must be distinguished from something else if it is to be experienced. Thus, the distinguishment of the present must be distinguished from the past if the present is to be distinguished. Therefore, there must be a change from one experience to another- a change from past to present. But, the experience of the present, and the experience of the past, though they must be different, must also be the same in some respect. For if they were not the same in some respect, then no comparison of them could be made, and therefore, no distinguishment of them could be made, and therefore, no experience of them would take place. We see here one irony of human existence/experience: It depends upon things being both different and equal in some way. There is still the question, however, as to whether it is necessary that any experience needs to be of something quantitative. For, if it is necessary that all experiences must differ from each other in some way, yet also be equal to each other in some way, why could it not be that all experiences were not equal to, and did not differ from, each other in the second indicated way- that is, the qualitative way? Could we not say that successive experiences could be the same in that they are qualities of experience, but differ in all other respects? Could not we say that each successive experienced thing in time could be qualitatively different than each other? The answer is that there *must* be a quantitative aspect to all experience, for only quantitative characteristics admit of *continuous change*. For example, if we could imagine that our experiences only ever changed qualitatively, then we would need to admit that these changes would be instantaneous, since it is inconceivable that one quality could continuously change into another. But, since to experience anything at one moment there must be a change from the previous, if all change of experience were instantaneous, then all experience would be instantaneous, which is the same saying that no experience would occur.

Put in other words: qualitative changes must occur instantaneously; and only quantitative changes can occur over an amount of time. Thus, since we admit that experience is dependent upon change, then, if we were to say that no quantitative experience were to occur, then we would say that no experience occurred over any amount of time (which is to say that no experience occurred). Alternatively, since we admit that experience is dependent upon change, then, if we were to say that only qualitative experiences occurred, then we would say that all experience would occur instantaneously, which is to say that no experience occurred over any amount of time (which is to say that no experience occurred). Indeed, it is conceivable that qualitative changes in experience can occur, even instantaneously. However, those instantaneous qualitative changes must be separated by intervals of continuous quantitative change if any experience is to be conceivable.

In total, then, we have four interdependent but distinct principles of cognition necessary for any conscious experience to occur: Oneness, Differentiation, Relation, and Equality.[11]

The Principle of Continuity

A thing must be finite if it is to be conceived as one individual thing. This implies that all things have boundaries, or, a beginning, and an end. But, if the beggining and the end of something are two distinct things, what is it that which connects the beginning and the end of one thing? The expected answer is: the middle. But, if the beginning, middle, and end of something are all distinct individual things, what impels the mind to comprehend them all in a unity? If every one thing that is conceivable has distinctly identifiable parts which are themselves conceivable as one, then, how does the mind choose what individual things will be included as parts of some other one thing which it recognizes? I contend that the principle necessary to the recognition of many things as one thing is the principle of continuity.

Continuity, most basically, means to indicate that over the entire course of any experience, something will remain constant, or invariant. The most readily understandable and fundamental

[11] Johannes Kepler provided some arguments for the existence of other universal forms, or principles, imbedded in the mind. These principles were similar to the principle of equality in that they acted as standards of judgement of relations, or comparisons. He called these forms "harmonies" and presented evidence of their existence similar to that of the Pythagoreans. For example, the mind can finds certain relations of sound to exhibit a distinct quality of harmony, while other relations of sound do not. Kepler attempted to establish a method of discovering all the harmonies based on what "regular" figures could be constructed with only definite statements about straight lines and circles- or, in other words, what numerical divisions of a circle's circumference could be produced using definite statements about straight lines and circles. However, the harmonies, even if their existence were admitted as real, do not bear on the logical edifice of mathematics in the same way as the other principles do.

instance of continuity is the continuity necessitated by the very concept of experience itself- namely, the continuity of the experiencer. Over the entire course of any experience, there must be present an experiencer. Only by the continuity of the presence of the experiencer is any experience conceivable. Thus it may be said that continuity is also a fundamental aspect of conscious experience and cognition.

The ability of the mind to distinguish individual things depends upon the principle of continuity. For each individual thing has individual parts, and so there must be a continuity of some common characteristic, or invariant, which the mind utilizes when it connects the various parts of something, and comprehends them as subsumed in a single unity. There must be some evident continuity over the course of successive separate experiences if those experiences are to be comprehended as the parts of a larger unity. The distinguishment by the mind of individual things is reliant upon the termination of some evident awareness of continuity. That is, it is by the termination of some identified continuous quality in experience which constitutes the meaning of "the end" of any one distinguishable thing.

If we try to conceive of a color which is not continuously extended over the entirety of some portion of the visual field we cannot do it. Therefore, the perception of color is inconceivable without continuity. If we try to conceive of a sound which is not continuously extended over the entirety of some portion of time, we find that it is impossible. Therefore, the perception of sound is inconceivable without continuity.[12] Clearly, no phenomenal quality of any sort could be experienced if it were not continuously experienced over some extent. It could be asked: If there were no continuity of quality over any extent of conscious experience, then what could the mind possibly recognize as any single thing to be experienced?

The principle of continuity finds expression in more than just those experiences in which there is a continuity of sense-perceptual qualities like color or sound. That is, the principle of continuity can be identified in instances in which the unity identified by the mind has no sense-perceptual quality which is continuous from beginning to end. Such is the case when the mind is able to recognize some other kind of quality, or characteristic, which is invariantly associated with the successive perception of separate sense-perceptual phenomena which are to be understood as different aspects of one single thing. For example, in a piece of music, there is no continuity of sense-perceptual phenomena. That is, there is no continuity of sound which the mind perceives throughout the experience of the piece.[13] For, even though there is much sound in a musical piece, it comes in the form of distinct and different notes, some of which are separated by long intervals of rest (no sound). Yet the mind recognizes a continuity from the beginning of the piece

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[12] This is not to say that the colors or sounds so continuously extended must be homogenous over the extension.
[13] Unless, of course, it were some silly noise in which all the notes were blended/slurred together with no rests. But no one would ever consider noise like that to be music.

to the end of the piece- even if the quality which is recognized as continuous is not immediately explicable.

The principle of continuity can be used to define aspects of our experience just as equality or any of the other principles can. Mathematics relies heavily on this principle in both the perceptual and non-perceptual way- in geometry as exemplifying the former, and arithmetic exemplifying the latter.

The Function of Memory

A consideration of the principle of continuity reveals to us another fundamental aspect of cognition without which no distinct experience could take place- memory. It is easily observed that since the present lasts only an instant, then, if there were no memory, all experience would occur in one instant, which is the same as if no experience were to occur.[14] Further, as said before, in every distinguishment of a thing which the mind makes, there must be distinguished a beginning and there must be distinguished an end. But the mind is capable of consciously distinguishing only one thing at a time.[15] Therefore, if the beginning of something were distinguished before, and then, at a later time, in the present, the end of that thing were distinguished, the only way in which the mind could relate the thing it presently distinguishes (the end) to the other thing which it distinguished before (the beginning), is if there were something which remained in the mind of the first thing distinguished which could be related to the last thing distinguished. Every distinguishment by the mind must irreversibly change the mind to this effect. As Bernard Reimann said: "With each simple act of thought, something enduring, substantial, enters into our soul." It is impossible to conceive how anything could be experienced without memory.

It will be noticed that memory is not here referred to as a principle, even though it has been demonstrated as a necessary aspect of cognition. This is because it is not a concept which applies to the description or definition of phenomena in the same way as the other listed principles do. The reasons for this will be made clearer in what follows.

[14] For example, if someone were to have absolutely complete memory loss of all experiences prior to one week, then their entire life would seem to last only one seek to them; for absolutely no memory of things prior to one week earlier could affect their current experience, including their sense of time. Similarly, if someone lost absolutely all memory of things occurring one day before the present, then their entire life would be experienced as long as the span of a single day. And so on. (Here we needed to say *absolute* memory loss, because even if there were a loss of conscious access to memories still in the mind -and this were called "memory loss"- those memories would still inform, or affect, current conscious experience, including the sense of time).

[15] At least consciously. There is evidence that the mind is capable of individually distinguishing multiple thing at once.

Although there may be more fundamental principles upon which cognition depends which have not been here enumerated, we will proceed no further in this vein. For even if we have not here enumerated all of the principles which are required for utilization in mathematics, an illustration of my method of treatment of the foundational problems of mathematics will nonetheless be afforded by the principles already identified, namely: Oneness, Multiplicity, Equality, Continuity, and Comparison.

II. <u>On the Reduction of Arithmetic</u>

Briefly, I will touch upon the issue of the formal-logical reduction of arithmetic. It appears here in the course of this paper as a sort of interruption of a train of thought, but, I could not find anywhere better to put it, and I felt compelled to give my brief thoughts on the matter since it is generally considered so important (or at least was- before Godel dashed the hopes and obviated the torturous labors of those who had been engrossed in it).

Admittedly, I was surprised to read that, after mathematicians had evidently succeeded in reducing all mathematics to basic arithmetic in the early twentieth century, Hilbert thought that it would be a task worthy of perhaps millions of man-hours of intellectual toil to find a new formal system, based on even more primitive notions than that of number, which would provide the formal-logical basis for the laws of arithmetic.

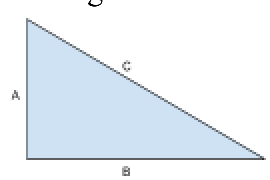
The laws of arithmetic seem to hold the rest of mathematics in place. But the meaningful concept behind formal arithmetic is "number", and it seems that no logical system could be conceived without presupposing the concept of number itself. For, if we try to build a formal logical system, it will have a certain *number* of axioms, and a certain *number* of postulates, and a certain *number* of rules of inference, and a certain *number* of distinct substantive notions which provide the system with meaning and relevance, and a certain *number* of all the distinct things which are characteristic of such a system. Thus, the attempt to provide the basis for a concept (and its logic) by another system which presupposes that concept in the first place seems to be somewhat silly. Needless to say, the process of reduction of notions and their logical systems to ones more primitive and encompassing must end somewhere- else no system could be established, and no distinct concept conceived. The concept of number will be discussed more later. But, for now, it can be said that the impossibility of creating a formal system without presupposing the notion of number is a good indication that the concept of number is an acceptable place to stop the process of reduction- and that the validity of formal arithmetic can only be measured against our ability to think about numbers with proper judgement.

III. The Generation of Logical Systems

In order to clarify what "logical systems" actually are, and how they are generated, the following point must be made and illustrated: It must be understood that there are, generally, two ways in which a person arrives at conclusions which they consider as implied by a given set of postulates, facts, or conditions. The first way in which this occurs can be called formal, or systematic. The second way in which this occurs can be called intellectual, or rational.

The Formal Method

The formal method is characterized by the adherence to a set of rules as to what statements are allowed to follow other statements.[16] That is, given a set of postulates, the formal method provides the rules which determine what conclusions are derivable from the postulates. In teaching and utilizing the formal method, no reason need be given as to why the rules of the system are what they are, and no consideration as to whether or not the rules lead to valid or true results -or even whether the rules are in any way meaningful- need take place. The formal method only specifies procedures which are to be taken given a set of conditions. This method, if it can indeed be called a method at all, is a form of animal behavior which humans can learn to employ through various kinds of training- just as animals. As an example of a utilization of the formal method of arriving at conclusions, take the case of this simple mathematics problem:



If A=2 and B=7 what is C?

The mathematics student is trained to answer this question by the following formal procedure: 1.)Find the number which is defined as equal to A multiplied by A. 2.) Find the number which is defined as equal to B multiplied by B.[17] 3.) Find the number which is defined as the number found in step 1 added to the number found in step 2. 4.) Find the number which when multiplied by itself is defined as the number found in the step 3. 5.) Finally, equate the number found in step 4 to C.

Thus, someone who does not know whether the procedure -or the Pythagorean theorem which the procedure is an expression of- is true or not, can arrive at the "correct" answer to the problem simply by following the formal procedure. Not only that, but the person doing the calculation does not even need to know whether the numbers defined as equal to "A multiplies by A" or "B

[16] Including, of course, symbolic statements.

[17] Of course, these steps are formal procedures as well. That is, the formal procedure called "multiplication" means looking at a multiplication table and repeating what is on the right side of the equation. Of course, most people are simply trained to memorize the multiplication table for commonly used numbers.

multiplied by B" are necessarily the correct numbers. Neither is it necessary that the word "multiplication" has any meaning to this person. The same can be said of the procedures of addition which was made use of. In fact, none of the symbols used in the procedure even need to mean anything at all to the person who carries it out. Even a blind person who had never experienced a visual magnitude (as in a visual triangle or visual line) in their life could carry out this meaningless procedure, with no concept of what were the nature of the magnitudes which the procedure is supposed to apply to.

As will be discussed, the formal method is extremely useful, but it is important to be clear about exactly what it really is. If formal systems are meaningless constructs, how is it that some formal systems are accepted as legitimate, while others are not? What makes a formal system distinguishable from any other formal system if they are all essentially meaningless? How is one to *know* whether or not the formal procedure they have adopted in the effort to draw conclusions is pertinent to answering real questions and solving real problems pertaining to distinct meaningful things? The answer lies in the *meaning* of the *subjects* which the formal system is *intended* to correspond to, and the ability of the mind to make judgements respecting the relations of those meaningful subjects. The answer lies in the intellectual method, or reason.

The Intellectual Method

The other way in which a person arrives at conclusions which they consider as implied by a given set of postulates, facts, or conditions, is by the intellectual method. It relies upon the fact that the mind has a tendency to recognize certain things as true or necessary based on the admission of other things. This is what the word "reason" properly refers to (at least, this could be said to be one aspect of reason). Ultimately, all formal systems - whether of logic, mathematics, or science- must find their justification in the ability of the mind to recognize the truth or the necessity of the elements of those formal systems on the basis of a reasoned consideration of what those formal systems are supposed to apply to. This is the case respecting the axioms of a system, and the rules by which theorems are deduced from the axioms (the rules of inference), and all other pertinent characteristics of the formal system.

That is, to say, the *only* basis for establishing any formal logical system is rational human mental activity. You must think before you formalize, not the other way around. Else, how were we supposed to *know* whether or not the conclusions derived by a formal logical system from given propositions were correct?

As an illustration of the way in which the intellectual method of arriving at conclusions from a set of postulates is utilized, consider the way in which the above referenced Pythagorean theorem was conceived of, and how the mind is able to identify the general validity of that theorem for all right triangles.

The Relation of the Formal to the Intellectual Method

To illustrate the relationship between the formal and intellectual methods, let the question be asked: “Why do humans create logical systems?” Obviously, the mind has certain tendencies which impel it to consider one thing as true and another thing as false. What are the things which the mind is capable of considering as true or false? Those things are *surmises*, or *conjectures* about subjects which the mind distinctly grasps. For example, the human mind has a distinct concept of color, or sound. These things are, to the human mind, distinct subjects. But the mere subject of color or sound considered abstractly, will not impel the mind to make any inferences.Thus, something needs to be *surmised about* the subject under consideration. These conjectures require a certain degree of definiteness if they are to lay the basis for further inference by the mind and thereby be judged as true conjectures.[18] Thus, the conjectures about a subject must make use of concepts whose meaning is clear, definite, or absolute.[19]

In the process of the mind making inferences from given conjectures respecting distinct subjects, certain regularities or generalities may be observed. The identification of a regularity or generality in the process of inference under a set of given conjectural conditions is the first step towards establishing a logical system, for such regularities can be abstracted, taken as rules, and formalized. The formalized rule can now be utilized to enable a person using it to arrive at an inference which *would have been* arrived at by someone utilizing mental reasoning in forming a judgement about a conjecture. For example, the mind is able to readily infer that if a thing A is equal to another thing B, and that other thing B is equal to yet another thing C, then C will be equal to the first thing A. Every time the mind is confronted with this situation of three equal things, it will always make the same inference based ability of the mind to grasp the nature of the concepts involved, like equality. This can be restated as a formal general rule like so: If a=b and b=c then c=a. This rule can now be applied formally to situations in which it might be useful to make a correct inference without the need for the person utilizing the procedure to think about anything. Take as another example the above referenced pythagorean theorem. Given a right triangle, the mind is able, through a process of reasoning, to make inferences as to the

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[18] That is, it seems to be part of the very essence of what a conjecture is that it is definitive in some respect. It may very well be the case that all self-conscious reflection requires a definite quality. The most basic act of self reflection perhaps being “I am experiencing this” which includes the definite “am”- or “is”. More will be said on this later.
[19] I would add that it is not only conjectures, but hypotheses which the mind considers. But, the difference is not one of substance, only of regard. For all conjectures are hypothesis. I don't know if there are any known technical-semantical distinctions between the two words. But, the difference, as it seems to me, is that a conjecture seems to have the general meaning that the conjecture is believed to be true, whereas a hypothesis is a supposition which is not necessarily believed to be true.

relationships which must adhere between the squares built upon the sides of that triangle. It finds that the square built upon the hypotenuse is as big as the square whose side is the other two sides (the base and the height) of the triangle combined. The mind recognizes this to apply to all instances of a given right triangle and formulates a rule as "In a right triangle, the area of the square of the hypotenuse is equal to the area of the squares of the two smaller sides added together.", or using symbols like $a^2 + b^2 = c^2$. Now, this rule, as mentioned, can be used to enable someone to *systematically* or *formally* find the information which could before only be found through a process of thinking and reasoning about geometrical figures.

These formal rules are useful, for they save time and mental effort. For example, someone who uses the formal pythagorean rule could solve many more problems of its applicable type in a given time period than someone else who did not make use of the rule, and, rather, proceeded to treat the problems by reasoning about the geometrical figures directly. For this reason, it is appropriate to call properly crafted mathematical rules -or similar kinds of formal procedures- "technologies". For the principle of technology generally indicates the capability of something to enable the person utilizing it to accomplish the same thing in less time, or with less effort. Additionally, these formal technologies can enable a person who utilizes them to identify valid inferences for sets of conjectures whose complexity and length would normally make such inferences extremely difficult or impossible due to the prohibitive limitations of our capacities like memory. That is, properly crafted formal systems can allow people to solve problems which were to complex to solve without them. Expressed one way, the legitimacy of a formal system used in this way is that it reveals, by purely meaningless formal procedure, what the mind *would* infer on its own *if* that mind were confronted with the same *sequence of conjectures* treated by that formal system.[20] David Hilbert expressed his own thoughts on the relation of formal systems to the mental process of reason in reference to his own formal systems as follows: "This formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. [...] The fundamental idea of my proof theory is none other than to describe the activity of our understanding, to make a protocol of the rules according to which our thinking actually proceeds"

It may be asked how it is possible to utilize formal procedures in place of the reasoning process of the mind. Is there not some doubt as to the fidelity of such formal systems respecting their correspondence with what the mind *would* judge if it were given these conjectures for its

[20] Here I say *sequence of conjectures* because, generally, the human mind does not make judgements about things in the stepwise fashion which characterize formal logical systems. It is actually impossible to systematically account for *all* of the nearly infinite varieties and dimensions of mental processes which lead the mind to make judgements. Therefore, the key to formal systems is that they systematize regularities of mental inference in regard to very elementary propositions/conjectures, and only those which can be sequentially ordered such that the final conclusions one wishes to prove can be arrived at by such "bite sized" steps corresponding to the most basic judgements which the mind makes.

consideration? What if there were a case in which the mind would make an inference from a given set of conjectures which were different from those inferences necessitated by the formal system? Sometimes, it is true, the mind, upon direct consideration of a conjecture, will make inferences which are at odds with what formal procedures indicate should be the inference. Sometimes it is found that a formal logical system actually leads to contradictions (which are usually thought of as conclusions to be avoided). But, very often, it is found that the judgement was mistaken in the direct consideration of postulates, while the correct inference from a set of postulates was given by the formal procedure. That is, sometimes our judgement becomes clouded and we draw conclusions not on the basis of reason, but on the basis of something inadequate. Indeed, Hilbert's original intention in creating his formal systems was to create something so abstracted from substantive intuitive meaning, that the mnd would not be at risk of resorting to any intellectual (good or bad) capacity in the process of drawing out the inferences necessitated by a set of axioms and rules of inference about mathematical subjects. That is, meaninglessness was to be a safeguard against any faulty reasoning that might enter, even in formal manipulations. However, it must be remembered, that the *only basis* we have for judging something to be correct is the ability of the mind itself to make the inference using its capability of reason, or, the ability of the mind to identify the generality of a rule by which inferences are derived- there is no other way to distinguish a correct inference from an incorrect one. Therefore, if the formal system is found to render conclusions which the mind is able to clearly understand as untrue, it is not always a result of bad judgement on the part of the mind, it could very well indicate that the formal system itself is lacking in meaningful correspondence with the subjects it is supposed to represent. Exemplary instances of this can be found in the history of science.

Dangers of the Formal Method

There are dangers inherent in the formal method. The most sever danger is misunderstanding what formal systems are in relation to reason. Sometimes -especially in cases in which a person has been subjected to intense formal mathematical training- a loss of the ability to distinguish between rational understanding and formal memorization can occur. For example, someone might claim that they "know" how find a solution to a problem in mathematics based on their familiarity with the formal procedure which (supposedly) furnishes the solution, even, at the same time, without actually knowing whether that formal procedure actually provides the correct solution. For example, the majority of students who utilize the pythagorean theorem would claim that they "know" what the solution to a problem involving its utilization is, yet, most in that same majority would not actually know whether the Pythagorean theorem is true, for most would have never performed the rational proof of the pythagorean theorem for themselves. This situation is prone to arise, especially since there is the "positive feedback" of hearing that one has obtained the "correct answer" by the authority figures which the student has chosen to defer to, whether that be a teacher or a text book, or something else. In the worst cases, the ability for rational thinking is actually lost -or never developed- as a distinct subject to conscious awareness. In such a case, the reasonable capacity may still be utilized, but without a self consciousness of the fact that it is being utilized.[21]

It is true that, in many instances, the performance of mathematical calculation involving formal procedures cannot be said to be purely formal or purely intellectual. The skilled mathematician may, in the course of a process of formal manipulations carried out for the purpose of solving a problem, alter his mental processes towards intellectual interpretation of the statements which are being made about the subjects he is studying; that is, the mathematician might start thinking about what substantive meaning and rational implications the statements produced by meaningless formal manipulation contain. Based upon such consideration of the conceptual content, the person performing mathematics may come to a recognition of a certain direction to proceed with the argument or calculation, and thus proceed with the formal manipulations towards the result desired.

This is not to be confused, however, with the sense of "instinct" which people trained in mathematics experience in the formal manipulative process. That "instinct" is not qualitatively different from the instinct of an animal which guides it to obtain a desired object by some set of actions- sometimes even relatively complex and roundabout ones: like a mouse in a maze, or a dog performing surprisingly complicated actions to obtain treats for example. That is, the "instinct" which guides a person to perform a certain formal manipulation in search of a solution to a problem is an effect of memory. The mind might remember that a certain formal statement is subject to manipulations which can lead one to the currently desired result after some additional steps. The person might not consciously be thinking that at the moment, and yet, still "feels" a sort of impetus to manipulate the current formal statement into the one that they, at some point prior, had manipulated into a form which is the one currently sought.This kind of skill in mathematics is thus obtained when much activity is performed in formal manipulations by a person in which the trial and error process and its results are registered in the person's memory. Obviously, a person who has a memory which more readily stores and makes available to conscious activity the results of these trial and error efforts will demonstrate greater skill in formal mathematical work.

I stated above that the mental sense of familiarity with a formal procedure or convention, and "knowing" by reason, can be mistaken for each other if the proper educational approach is not

[21] The question may be asked: can anything which is not self conscious be called "reason". If so, must we not consider reason itself as similar to a sort of physical principle which mankind may unconsciously utilize, but which becomes of much greater use to man once he is made conscious of its existence and consciously utilizes it for his benefit?

adopted.[22] Similarly, there is a danger posed by improperly situated formal training of mathematics to the development of a distinct self awareness of the process of reasoning within the mind of the student. For, the student might mistake the above referenced "instinct" for the process of intellectual/reasonable judgment and inference. That is, they may come to mistake the mere instinctive impulse constructed by stored memories of trial, error, and reward, (which governs their behavior in a particular situation) for the process of reasonable judgement itself. Such an unfortunate state of affairs, can, in extreme cases, lead to a kind of "savant syndrome", in which the individual loses confidence in their ability to think about or discuss any subject in which they do not experience the "instinct" which leads them to respond to given statements. For this instinct, as we have demonstrated, is a product of memory obtained by formalistic trial and error work in some field of investigation. Thus, if the person has not discoursed in a field of investigation to an appreciable extent and received the authority figure feedback which teaches them what is "right"- what are the formalities of exchange, the conventions of discourse- then the person will not experience the "instinct" which hey mistake to be the substance of judgement and reason, and will, thus, shy away from such discourses, and focus more and more on their area of special study. The same problem can be expressed in a different form in the case of the individual who, mistaking the "instinct" for rational thought, proceeds to learn (or train themselves) in the conventions of some particular subject. They observe the patterns of statements and responses within the course of discussions of a particular subject, and, based on who the individual selects as the authority figure, builds up a store in their memory of statements, lines of statements, or kinds of statements, which, if produced are to be met with responses of a particular rewarding type from that authority figure. This memory store expresses itself as the "instinct" to respond in a certain way given the encounter of statements in

[22] Indeed, it is on this very point that the national security and strategic implications of proper educational practice are put into clearer perspective. If the educational practice of a nation is of a sort which does not enable the students, -and thus the general population over years- to distinguish between knowing and mere familiarity with conventions, then the population in question is, to that degree, more susceptible to mass manipulation by whomever controls the channels of media, political discourse, religious institutions, university curricula, or any other "authority figure" institution. That is, the more the "authority figures" of society will be able to control the masses of people in society through the manufacture of conventions. Today, as Bertrand Russell's horrifying statements on the issue would lead us to suspect, it seems that this point has not been missed by the oligarchy of Europe and the US, as changes in educational practice in the US reflect this trend. That is, it seems like some people are taking the following recommendation of Bertrand Russell to heart: "Education should aim at destroying free will so that pupils thus schooled, will be incapable throughout the rest of their lives of thinking or acting otherwise than as their schoolmasters would have wished. . . . Influences of the home are obstructive; and in order to condition students, verses set to music and repeatedly intoned are very effective. . . . It is for future scientists to make these maxims precise and discover exactly how much it costs per head to make children believe that snow is black, and how much less it would cost to make them believe it is dark gray...When the technique has been perfected, every government that has been in charge of education for more than one generation will be able to control its subjects securely without the need of armies or policemen."

discussions on the subject so “learned”. Needles to say, adherence to this kind of behavior precludes, absolutely, actually creative or original thinking by the individual in the domain for which they have so trained themselves.

Another danger attending the utilization of the formal method is the possibility of producing meaningless statements on the basis of the rules of the system. That is, based on the rules of the formal system, statements may be produced which have no capability of being interpreted in terms of the substantive notions which the formal system is ultimately supposed to relate to.[23] This is especially prone to occur when symbolic language is used in the formal system. For example, utilizing the formal procedures of algebra, the following equation $x + 1 = 0$ seems to be solvable by formally inferring that $x = 0 - 1$, or $x =- 1$. But what is “-1”? What is a quantity that is less than no quantity? From the standpoint of magnitude and collections of things (numbers), the symbol has no meaning. This is not to say that new formal rules can be established which govern the manipulation of these kinds of symbols, and new meanings attributed to them. Take, for example, R.W. Hamilton’s view of the matter: “It must be hard to found a *science* on such grounds as these [notions of negative number], though the forms of logic may build up from them a system of expressions, and a *practical art* [emphasis mine] may be learned of rightly applying useful rules which seem to depend upon them.” An example of a “practical art” which put to meaningful use the formal rules of negative numbers, is the scheme of John Wallis in denoting units to the left of an arbitrary point on a cartesian grid as “negative” and units to the right as “positive”. Another “practical art” would be that of dealing with monetary debts and assets.

As another example of the danger just referenced, formal algebraic manipulation may lead one to conclude that there is a number represented by the symbol $\sqrt{-1}$ by formally solving the equation $x^2 + 1 = 0$. However, despite that such symbols may be used formally, and new rules might be established for them, a new suitable substantive meaning for these symbols must be produced if -to say the obvious- they are to be of any meaning for us. The meaning of this symbol, in fact, as well as the symbols for “negative numbers”, were subjects of great uncertainty for mathematicians for quite some time. Take the following quote from a letter written by Gauss in 1825 “The true sense of the square root of -1 stands before my mind fully alive, but it becomes very difficult to put it in words; I am always only able to give a vague image that floats in the air.” In other words, neither Gauss, nor anyone at that time (that I am aware of) had conceived of any substantive meaning for the symbol $\sqrt{-1}$. That is to say, it was still a meaningless symbol on a piece of paper produced by formal manipulation. However,

[23] Ideally, formal systems are supposed to contain all the rules necessary for the prevention of this situation, however, as we will discuss later, no complete formal system has ever been found to encompass basic mathematical notions like the operation of numbers (arithmetic), and in fact, Godel proved that none could be found.

Gauss later put forward a practical way of interpreting the symbol on a geometrical basis in 1831. That is, it took years for the most brilliant minds in the world to come up with a suitable interpretation of the symbols like $\sqrt{-1}$ which was intelligibly related to the results of the formal rules of manipulation which gave birth to such symbols in the first place. We will discuss this kind of development in mathematics later.

In the worst cases, meaningless symbols generated by faultily constructed formal systems might lead people on an endless chase for the discovery of profound metaphysical meanings which lie hidden from view, but which are evidenced by such symbols. This kind of mania is reinforced by the fact that the formal system which furnished such meaningless symbols is of great use for solving meaningful problems. Upon furnishing a meaningless symbol with such a demonstratively useful and meaningful formal system, it may be believed that, since the formal system produces statements which are meaningful and useful in some cases, it must produce statements which are meaningful in all cases, and thus, the creation of a meaningless symbol is believed to be an indication that there must be discovered the hidden meaning behind that symbol. But this is not necessarily the case. This is not to say, however, that sometimes the meaning of the results of formal manipulation are not readily perceived and that people should not seek for meaning if they currently attribute none to what they produce by formal procedure.

Indeed, though after years of strenuous mental effort new interpretations might be created for the otherwise meaningless symbols resulting from formal manipulation, it must be considered as a possibility that there may be no useful new meaning possible to be attributed to a symbol generated in such a way. If reinterpretation does take place, however, any such reinterpretation of the otherwise meaningless symbols of formal production necessitates that the entire symbolic formal system must be reinterpreted to accord with those new meanings. For example, if we encounter the negative numbers resulting from formal manipulation, we have choices as to what to do: On the one hand, we might simply say that it is impossible to take away a greater quantity from a lesser quantity, and therefore maintain that the formerly produced symbol of -1 is meaningless. This decided, we would proceed, perhaps, to modify the formal rules of our algebra to avoid production of these meaningless symbols in the future. A rule in the formal system which mandates, for example, that a larger quantity cannot be subtracted from a lesser would do the trick. On the other hand, we might try to find some new meaning to attribute to the formally generated symbol of -1. We might, for example, adopt the approach of John Wallis, and interpret the “negative numbers” to be symbols for a certain set of lengths on a cartesian plane which lie below and to the left of an axis origin (zero point). However, if we do this, then we must reinterpret all the symbols of our formal system according to this meaning. That is, even the symbols like 1, 2, 3 etc. must be *reinterpreted* as magnitudes in a certain graphical configuration, as opposed to their original interpretation as *numbers*. All of the formal rules of the system must now be proven as valid based upon the consistency with which they adhere to the characteristics of the *new* interpretive meaning. That is, all the formal operations must be interpreted as corresponding to the operations and relations of the new subjects which the formal symbols are now taken to represent. For example, the operation of "multiplication" of two symbols A and B has a very definite meaning if we interpret the symbols A and B to represent abstract collections of things (numbers): the meaning of multiplication of one collection A by another collection B is to substitute each member of B by the collection of A to form a new, larger collection (called the product of multiplication). But if we take the formal system constructed for the way we understand numbers to work and reinterpret it as corresponding to graphical relations of points and lines, then the meaning of multiplication changes to some operation which we may perform with these subjects. The same can be said of Gauss' reinterpretation of the formal system of numerical algebra as corresponding to a units of length in a cartesian plane, the characteristics of which are probably well known to the reader as the "complex plane".

The distinction between the current and prior meanings of the symbols must not be lost. Great confusion can result if one forgets that they have changed the meaning of the symbols in a formal system to justify a certain result of formal manipulation, or to find a new useful meaningful system. For example, new meanings were attributed to the symbols in the formal system of algebra to justify the formal generation of entities like -1 and $\sqrt{-1}$. But, then, many people forgot that the meanings of the symbols had been changed, and thus they claimed that they had discovered new "numbers", when, really, they did not. The concept of "number" had to be abandoned as the meaning of the symbols of the formal system, and new concepts had to be introduced as replacements. Thus, to say that we "discovered new numbers", in this way, is ridiculous.

The final danger inherent in the formal method is the possibility of faulty or incomplete construction of formal systems. This is the danger which has been the greatest focus of attention of the mathematicians involved in the "foundational" problems of mathematics for the past dozen decades. If the formal system contains faulty rules, it may lead one to formally derive inferences which are not true or contradictory.[24] Additionally, if the formal system does not contain enough rules, it will not capture all of the things which might be rationally inferred about the subjects under consideration. A well known example of the dangers referenced is the way in which it can be proven that 1=2.

1) $a = b$	1) Given
2) $a^2 = ab$	2) Multiply both sides by a
3) $a^2-b^2 = ab-b^2$	3) Subtract b^2 from both sides

[24] This statement is not redundant. See Nicholas of Cusa's work.

4) (a+b)(a-b) = b(a-b) 4) Factor both sides
5) (a+b) = b 5) Divide both sides by (a-b)
6) a+a = a 6) Substitute a for b
7) 2a = a 7) Addition
8) 2 = 1

This would be a formally valid conclusion in a formal algebra which did not include the rule that we cannot divide by 0.

Now that it is established that the mental process of judgement, or reasonable inference, proceeding from the mind's consideration of conjectures respecting substantive notions , is the basis for any formal system of logic[25] which can be constructed, let us proceed to look a little closer at what the mind requires in order to generate judgements about things.

Prerequisites of Reason

--*Distinct Concepts*
What is required for reason to take place? What is necessary to the process of the mind making judgements about things? Clearly, the most basic thing the mind needs in order to make judgements about something is *something*. The things which the mind is capable of judging are, obviously, of a conceptual nature- they are meaningful concepts, thoughts, notions, experiences, associations.

But distinct things alone are not sufficient to account for the capability of the mind to judge them. Indeed, animals have distinct concepts, experiences and meaningful associations.[26] For example, animals experience distinct colors, or hues, of various kinds. A dog will even become ecstatic upon seeing his owner (which the dog is able to distinguish from other people) holding a leash (another distinct object which the dog can recognize), because the dog *remembers* the joy of going for a walk which always follows the sight of the master holding the leash. As we can see, the dog has distinct experiences, and is even capable of association, by memory, of different

[25] I understand that there may be a distinction between logic per se and formal systems of other types in mathematics. Logic seems to me nothing but a formal system which encompasses many different kinds of conjectures, (but not all) whereas mathematics is comprised of formal systems relating solely to certain mathematical notions, but also utilizing logic etc. Actually, I suppose the answer to this questions depends upon what foundational school of mathematics you choose to believe, formalists who used logical and mathematical axioms, or logistics who hoped to build all of mathematics on logical axioms, for example.
[26] It may be objected that it is not known whether animals experience anything as people do. That is, that animals may be nothing but highly complex biochemical machines. But, I do not think that animals are nothing but machines. I tend to think that the evidence supporting the notion that animals undergo qualities of experience similar to the qualities of experience which our own minds undergo is about as firm as the evidence supporting the notion that people other than ourselves undergo qualities of experience similar to our own.

elements of its experience just as humans are. Could we not go so far as to say that all of the fundamental principles of cognition and experience listed in the first section, must also apply to the mental experience of animals? If we admit that animals have experience, then it seems that we must. Thus, the dog has concepts and mental operations just as humans do, yet, the dog is not capable of judgement. What is missing?

--*Self Consciousness*

We can see that one missing element is self consciousness. For, though the dog may have distinct notions and mental operations, it is not capable of reflecting on the fact that it does so. If a dog becomes excited at seeing a leash, it is only due to an impulsive instinct ingrained in it's memory by repeated association. But the dog is incapable of reflecting upon this fact. Humans may have tendencies of association similar to animals, but they can be made self consciousness of this. For example, there is a fellow at the office I work at named Bob who makes very good meals. Thus, whenever I hear the words "Bob is making dinner for the office tonight" I become about as happy as a dog who just saw his owner pick up a leash- but, I am able to think about this occurrence and communicate it to others. Animals have distinct experiences and thoughts, but they cannot think *about* those distinct experiences and thoughts. Though they experience color, they cannot consider color itself as a subject of thought. Though they experience sound, they cannot consider sound as a subject of thought. Though they experience things in accordance with certain principles, they cannot become self conscious of those principles and their operation. In short, being aware of distinct things is not enough to account for rationality, one must be aware that they are aware of distinct things. If there is no capability to think about the distinct things which the mind recognizes, then there is no capability to make judgements about those things.

--*Meaning/Substantive Mental Content Corresponding to Terms in the Proposition*

Despite the obviousness of this consideration, it will be found later to be of importance when addressing some of the questions regarding the nature of mathematics. The point is, simply, that in order for the mind to make inferences (*cognitively*) respecting propositions, the propositions themselves must have distinct *meaning* to the mind which is confronted by them, and each part of the proposition must also have *meaning*. The words, sounds, symbols, or any other perceptible artifices utilized in the communication of the proposition must be understood to indicate substantive notions which the mind is conscious of. If any proposition is made whose individual terms mean nothing to the mind which encounters it, then the proposition is meaningless and no rational judgement can take place.

It is possible, however, that an individual may mistake the *sense of impetus* to deal with a proposition in a certain way as *meaning*. As discussed earlier, an individual may be trained to carry out a formal procedure given a certain kind of proposition even without having the slightest conception as to what the proposition is supposed to mean. In this case, such an individual will experience a sense of impetus, ingrained through repeated training, as to how to act in reaction to being confronted with the proposition in question. If such a person were then asked: "Does this proposition mean anything to you", or "Do these symbols in a mathematical equation mean anything to you?" they may very confidently respond that it is indeed meaningful to them, but this might only indicate that the *meaning* of the proposition to this person was the sense of familiarity with the formal procedures taken in reaction to the proposition, and/or the sense of impetus to such procedures.

--*The Fundamental Propositional Concept*
It will be noted here that the capability of the mind to self consciously reflect upon distinct concepts is always attended by the general notion of "is". The notion of "is" might be referred to as the fundamental propositional concept. The mind must be conscious of this concept if it is to make propositions and inferences. This concept is distinctly understood as indicating the definite characteristics, qualities, classifications etc., *of* a subject- things which are not considered the subject itself. It is evident that our notions of "is" and "is not" are derived from the basic nature of experience. For one either is or is not experiencing something. The concept of "is", or "being", is a notion which is inseparable from self-consciousness itself. For the idea of "is" seems to be implied by the very self-conscious notion of any distinguishable thing. For any distinguishable thing must be consciously considered as *being* some distinguishable thing.

--*Distinctness/Definiteness*
It seems clear that, in most cases, any distinguishable thing will either be or not be of one certain quality which constitutes it as distinguishable from other things. This, of course, depends upon the definiteness/distinctness of the quality which renders something distinguishable. Ambiguities as to whether any thing is indeed some distinct distinguishable thing of a certain quality are common in proportion to the indefiniteness/indistinctness of the notion of the distinguishing quality itself. For example, there is more room for ambiguity in the statement "The dog is walking." than there is in the statement "The dog is alive". For a dog may be limping, or hobbling, or crawling, or trotting and it is ambiguous as to where we should "draw the line" between these distinct yet similar and blendable notions, and it will be correspondingly more difficult to judge whether a dog really is or is not walking in any given instance. But the notion of "alive" is much more definite than the notion of "walking", and the distinction between alive and not-alive (dead) is about as sharp a distinction which the human mind is capable of making. Therefore, there will be little ambiguity as to whether a dog is alive or dead in any given instance.

Because the fundamental propositional concept "is" can be used to construct propositions which make use of any notion we may conceive, and, because the notions which the human mind grasps are of varying degrees of definiteness/distinctness, the inferences which the mind makes regarding propositions vary in their degree of certainty and regularity. For example, returning to the propositions above respecting the dog, if the mind were provided with the proposition "The dog is walking" then that mind might make inferences about the dog which are consistent with the notion of "walking", but, that mind might also make inferences about the dog which are consistent with someone else's notion of "trotting" or "hobbling", since it is not absolutely distinct as to where walking ends, and hobbling or trotting -things which are *not* walking- begins. On the other hand, if the mind were provided with the proposition "The dog is alive", few minds would make inferences about the dog which were consistent with the notion of a not-alive (dead) dog, for the notion of alive is very distinct from the notion of dead.[27] The possibility for variability of inference based on the indistinctness of the notions involved in propositions begs the question as to how we might identify the most absolute and distinct notions, so that we can make the most regular, secure, and certain inferences from propositions. It is only to the extent that a conjecture about something invokes an *absolute* notion that the conjecture can be said to be of definite import regarding inferences from it.

Absolute Notions

What, then, are the notions which the mind can grasp in such a way as to consider them of absolute distinct and definite meaning? What are the concepts which need to be invoked in order to provide the mind with the basis of making rational inferences which are identifiable as regular and certain?

As we have demonstrated, the mind operates on the basis of a set of fundamental principles, without which it is inconceivable that it could think. The above referenced capability of the mind to engage in self-conscious reflection of it's own experiences and operations of thought extends also to the fundamental principles themselves.[28] If the fundamental principles themselves are distinctly grasped by the mind, then the mind is in a position to consciously consider things in reference to to them. To the self-reflecting mind, the fundamental principles are so distinct to that they are considered as absolute. Because these concepts are so clear and unambiguous, and because they are demonstrably common to all conscious minds, it is readily believed that inferences from conjectures which invoke the fundamental principles will be the same for all minds which consider them. That is, inferences which are made on the basis of considerations of

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[27] It is important to keep in mind here that I am not referring to formal derivations of inferences from propositions as might be found in a categorical syllogistic framework which included the terms "dog" "walking" "trotting" etc., in which there would be no possible inferences deducible other than the ones allowed by the rules of the formal system. I am referring to the actual process of thought attending the mind's consideration of these propositions and their *meanings*. Specific formalizations of procedures of inference from propositions respecting indiscreet/indefinite notions can always be challenged by someone who disagrees with the formal construction based on their own experience with judgements respecting the notions under consideration.

[28] (Otherwise, of course, how would we be able to refer to them in this paper)

propositions invoking such principles will be said to be universally valid or true. We will see, later, how it is that the concepts of absolute principle lie at the foundation of most of the body of mathematics.

Logical Systems and Rational Modalities.

With the formal and intellectual methods by which conclusions are derived from postulates thus illustrated, we are in a position to clarify the distinction between a "logical system" and what I shall call a "rational modality". A logical system is a set of rules respecting the statements about some subject which are allowed to follow from postulates and statements derived from them. By the term "rational modality" I mean to signify nothing other than the way in which the mind reasons and intellectually generates inferences about a particular distinct subject/concept. For the mind reasons in different ways about different kinds of things. The same proposition made of one thing may lead (reasonably/rationally) to a completely different inference than the same proposition applied to another thing. For example, if we judge the proposition "A line divided in two results in two lines" we find nothing objectionable about it. If we judge the proposition "A worm divided in two results in two worms" we find nothing objectionable about it. But if we judge the proposition "A man divided in two results in two men" we find the statement to be objectionable. Therefore, the logical formula which might have been derived from judgements about the first two statements "Any (thing) divided in two results in two of that (thing)" is clearly not applicable to all subjects. The difference in the validity of the same proposition applied to different subjects is an illustration of the *discrepancy of the rational modalities* associated with different subjects, and the *incompatibility o f the corresponding logical systems*. Indeed, no logical formula of the type used in our example is universally applicable to all things (although some people try to convince themselves that the "three laws of logic" do).

To further illustrate the distinction between logical systems and rational modalities, let us examine the way in which regularities of inference are established on the basis of considerations of propositions utilizing the word "is". The word "is" has a different meanings, and, therefore, corresponds to multiple rational modalities. Thus, there will be different logical systems for propositions pivoting upon the word "is". The first meaning of the word "is" we will look at relates to the establishment of conventions in language- it merely indicates the interchangeability of different words (sounds), symbols, or other perceptible instruments of communication, to fulfill an intention to indicate one distinct idea/concept. For example, if the propositions were to be put forward: "A dog is a pooch. A pooch is a hound. A hound is a canine." one would easily infer, given the intended meaning of the word "is" in this context, that a pooch is also a dog, a hound is also a pooch, a canine is also a hound, a dog is also a hound and a canine, and a canine is also a pooch etc. For, in this instance, the word "is" only means to indicate that different names are intended for the same thing. The rational modality of this meaning of the concept “is” in the proposition is simple. Thus, a formal rule can be established which of the form: “All the terms in any proposition which lie on either side of -and/or in between- the term “is” are interchangeable.” However, to repeat, this formal rule will only apply in situation in which the word “is” corresponds to the meaning in which it was used in our first example. For example, consider the following propositions: “A pug is a dog. A dog is furry” Are we able to *formally* draw an inference based on the above established rule of inference respecting propositions involving the term“is”? That is, are we allowed to infer that “a dog is a pug”? Or “Furry is a pug”? No. Why? Because the nature of the relevant concept involved -that denoted by the word “is”- is different than in the previous example, and thus corresponds to a different rational modality, or way of thinking about propositions involving it. “Dog” is considered to be a *type*, or *class*, but, in this case, “is” means that the preceding term *is a member of* the class indicated by the next term. The notion of class, or type, and the concept of participation in classes or types, compels the mind to inferences, from such propositions, which do not include “commutability”- that is, the ability to say “B is A” given the proposition “A is B”. Thus, any formal procedures of inference respecting propositions of this type will be different than those established for propositions involving “is” used with the first referenced meaning in our first example. The rules would need to encompass the empirical psychological fact that the mind infers that “A pug is furry” but revolts at the idea that “fury is a pug”. Such logical systems have been developed, and they are referred to as categorical syllogisms.

Meaning

As indicated above, but reemphasized here, the rational modality, and derivative logical system, respecting any subject will depend upon the concept of the subject- will depend upon what the subject means to us as rational beings. Because of this, all inquiries into the merit of a logical system must include a careful consideration of what the concepts indicated by the language of the system (including symbolic language) actually are. Then -on the basis of a self-conscious consideration of the rational modality corresponding to those identified concepts- can a rational consideration of the viability of the logical system take place.

It has been reported that Hilbert once said that all of the terms of his formal geometry could be replaced with words like “beer mug”, “tables”and “chairs” and still be of the same validity. But this is true only if we view the internal logical structure of the formal system. The undefined terms of any formal logical system can be attributed any word and also any meaning we want. However, after doing so, we will need to judge whether or not the formal system actually makes any sense- whether the system actually captures the relevant relations of the substantive notions which have thus become associated with the system by the placement of certain words in the “undefined term” position of the system. For Hilbert's example, obviously, his formal system would still be logically valid if “beer mugs” and “tables” replaced “points” and “lines”.

However, the formal system would then become meaningless, since we don't conceive of beer-mugs and tables as having the same relations as points and lines. In other words, the rational-modality of the concepts of beer-mugs and tables is different than the rational modality of the concepts of points and lines, and, therefore, substituting the meaning of the undefined terms in a useful formal system of one for the other would render the system useless/meaningless.

Two Kinds of Generation

Based on what has been said up to this point, it can now be shown that there are *two* ways in which formal/systematic logical systems corresponding to meaningful subjects can be generated.

A formal logical system which is in correspondence with meaningful concepts/subjects can be built either
1.) *After* there has been chosen what meaningful concepts are to be provided with a formal system.
Or, 2.) *Before* there has been chosen any meaningful concepts to be provided with a formal system.

In the first case, certain meaningful subjects are considered, and then, a formal logical system is constructed which is adapted to how the mind perceives its own process of judgement -or rational modality- respecting those subjects. In the second case, a meaningless formal system is constructed for its own sake, or by accident, and, afterward, meaningful interpretations of the formal system are conceived. In the first case, the formal system is constructed to adapt to the meaningful subjects/concepts. In the second case, the meaningful subjects/concepts are thought-up to adapt to the meaningless formal system (and thus to make it meaningful). In the first case, substantive notions come first, and the formal system after; in the second case, the formal system comes first, and the substantive notions after.

As exemplary of the first case, take the way in which the formal rules of arithmetic and algebra were created to adapt to the rational modality of the concept of numbers. As exemplary of the second case take the development of the notion of “imaginary numbers”- as discussed earlier, it started with the formal production of a meaningless symbol- meaningless from the standpoint of the concepts which the formal system was intended to correspond to in the first place; but, afterward, a new meaningful interpretation of the formal system, including the meaningless symbols just mentioned, was created by Gauss. The same could be said of the “negative numbers”.

These two different kinds of processes of generation of logical systems feed into and build upon each other, even if by accident. The development of aspects of mathematics, as exemplified by the case of negative and imaginary numbers and their meaningful interpretations, has depended , to a certain extent, upon this interplay. As Gauss himself pointed out: “One should never forget that the functions [of a complex variable], like *all mathematical constructions*, are only our own creations, and that when the definition with which one begins ceases to make sense, one should not ask, ‘What is is’, but what is convenient to assume in *in order to make it significant*.” (Emphasis mine)

I would mention that a consideration of this point leads us to a clearer understanding of other aspects of the development of mathematical thought. The case of the development of non-Euclidean geometry is usefully referenced, and clarified in its own right, in light of this just elaborated distinction.

The parallel postulate of Euclid troubled mathematicians for hundreds of years. There was uncertainty as to its necessity or validity. Some recognized that if it could be proven that the substitution of the parallel postulate with another postulate led to contradictions, then the validity of the parallel postulate could be considered more certain. A fellow named Sacchieri attempted to do this. He created a new geometry based on a set of postulates similar to Euclid’s but which modified the parallel postulate in order to see if he could derive a contradictory result- something which would guarantee the invalidity of the new postulate. Saccheri carried out his plan and found that he formally/logically produced results which were very strange to him. These results were so strange that he thought that they must be contradictory, and thus he announced to the world that he had found contradictions in the geometries utilizing postulates other than Euclid’s parallel axiom. But Saccheri did not actually find any formal contradictions, and so his grand announcement was a false one. How is this related to the issue of the two ways in which logical systems are generated?

The postulates of Euclidean geometry are the formal devices comprising the formal system intended to correspond to the rational modality associated with the way the human mind thinks about “lines in space”. Simply put, the system of Euclidean geometry was built to correspond to our notions of visual space and the corresponding notion of a visually straight line. Those are the meaningful concepts/subjects which Euclid's logical system was constructed to adapt to. When Saccheri changed the parallel postulate of that logical system, the system lost its correspondence to the concept of “straight lines in visual space”. Thus, even though their were no formal logical contradictions in the results which Saccheri procured, the results were so meaningless from the standpoint of the concept of “lines in visual space” that Saccheri believed that his results amounted to logical contradictions. But there is a difference between logically contradictory and meaningless. What was missing then in the case of Saccheri?- for, as we see from the history of the subject, non-Euclidean geometry later became widely accepted.

The answer to this question lies in the above referenced distinction between the two processes of generation of logical systems. Bolyai, Lobachevsky, and Gauss all delved into the subject and reveled in the ways in which new geometries, whose formal logical consistency seemed just as secure as Euclid's, could be created by substituting various other postulates for Euclid's parallel axiom. A recognition had to have taken place among (at least some of) them that, indeed, the newly created systems of geometrical logic could be relevant to mankind in some way. But in what way were the systems meaningful? What substantive notion could be conceived which corresponded to the logical relations in their newly created geometries? The stunning realization came that the relations of *physical space* and/or the relations of *physical processes*, as opposed to the *phenomenal space and straight lines* of our visual field, could meaningfully correspond to the logical relations of the new geometries. Thus, a new, meaningful interpretation of an otherwise meaningless logical system (a "Type 2" generation) was the basis for one of the greatest revolutions in scientific thought. We see that Gauss, for example, attempted to experimentally verify a deviation from Euclidean relations in the physical process of the propagation of light, (which he evidently thought of as corresponding to "straight lines" in *physical space*)[29].

IV. <u>Space and Geometry</u>

At this point, having laid the basis for a discussion of the issue of space, I shall take the liberty to elaborate on what has been said immediately above.

The point which I raised about the meaning associated with the non-Euclidean logical systems may incline some readers to objection. The key to the problem lies in two considerations: 1.) A lack of distinction between the concept of *physical space*, and the concept of *phenomenal visual space*; and 2.) An appreciation of the fact that any formal logical system is only useful or meaningful if it is adapted to the characteristics of notions which are meaningful and substantive to us.

Phenomenal visual space is familiar to all of us who have vision. The self-evident gross-phenomenal subject of the two-dimensional visual field is the most basic aspect of our

[29] Gauss performed his experiment by measuring the angles of registration of light on three mountaintops.

notions of phenomenal visual space. Whatever we perceive in visual space is perceived on our two dimensional visual field. (The two dimensional visual field is subject to certain kinds of judgements, however, such as the judgement of depth.[30]) Without getting into the conceptual/philosophical issues of how we might try to define a line, or how what we conceive of as lines are generated, this much can be said: the notion of "line" is conceptually clear to us. The notion of "straight line" is also conceptually clear to us, as long as that term refers to a definite kind of phenomenon in our visual field (and its conceptual abstraction).[31] A "straight line" is only straight if it phenomenally manifests the principle of equality and continuity in a certain way (or if it is conceived of as doing so). Thus, any line that we encounter in phenomenal visual space or any line we conceive in our imagination, can be judged against our concept of straightness. Euclid's formal system of geometry was created to adapt to these meaningful concepts of visual experience and their conceptual abstractions like the notion of "straightness". Thus, we can see why many mathematicians, upon the development of the non-Euclidean systems, thought that such systems were meaningless, for, in fact, they were. That is, they were systems which were not adaptable to the clear concepts which the Euclidean system was designed to adapt to, i.e. the concepts of visual phenomenal space. Therefore, lacking another interpretation, they were, by definition, meaningless.[32] (To emphasize here briefly: Whether these concepts, derived from the characteristics of visual experience, correspond to the characteristics of a hypothesized reality outside of our experiences is another question entirely.)

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[30] The judgment of depth is supported by multiple influences, such as our memory of our ability to alter the phenomena of our visual experience through bodily action, as well as the binocular visual effects created by the use of our two eyes. The judgement of depth is a relatively automatic operation of the mind, not requiring any conscious efforts on our part. Because of this, it can be mistakenly thought that the quality of depth found in our visual experience is divorced from our process of judgement.

[31] We might say that the notion of "straight line" *means* a visual line the surfaces bounding which are not concave/convex to each other. As far as I can tell, Kant was correct in his assertions that Euclidean geometry is inherent to our notion of space- if our *meanings* of the word "space" and "straight line" are derived from our phenomenal visual experiences. For this reason, I would point out that it is a common mistake (also made by Kline) to claim that Kant was wrong in asserting that Euclidean geometry is inherent to our notions of space. For Kant meant by "space" only the phenomenal space f our experience and the notions associated with it- not any hypothetical space in a reality outside our experiences (a reality which he claimed we have no access to whatsoever).

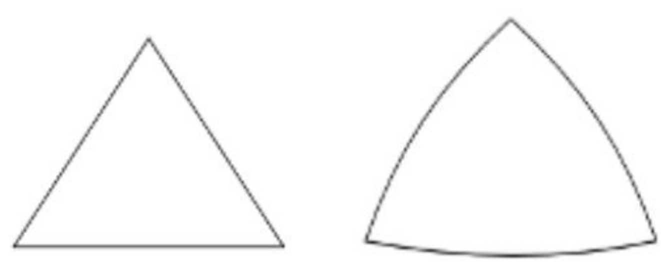
[32] However, I would never claim that all of the opponents of non-Euclidean geometry opposed it for legitimate reasons. I would also add that those who developed the non-Euclidean systems likely had ideas, or at least presentiments, of new possible interpretations of these formal systems which drove them to accomplish the work they did, even while in a hostile intellectual environment. In fact, the above referenced quote from Gauss respecting his own emotional/intellectual presentiment respecting the creation of a meaningful system for the formal entities of -1 and *i* , provides us with a glimpse into the state of mind of the scientist in such periods of creative intellectual tension. It may be of great benefit to scientific education to include in the curriculum readings of the letters of correspondence and private diaries of the great scientists from the intervals of time immediately preceding their fundamental scientific discoveries. By familiarizing those hopeful to become contributors to mankind's knowledge with the intimate psychological, emotional, and rational processes experienced by revolutionary scientists, as they are expressed in such letters and diaries, those scientific hopefuls can become better acquainted with what is truly involved in creative work, and thus become more likely to make genuine unique contributions themselves.

For example, the attempt to demonstrate the feasibility of the logical soundness of the non-Euclidean geometries included resorting to a reinterpretation of “straight line” to any line, curved or not, in visual space. Models using curved lines were used in 2 and 3 dimensions to illustrate the logical soundness of the non-Euclidean formal systems. For example, Beltrami recognized that double elliptic non-Euclidean geometry was not only susceptible to modeling within a two dimensional plane, but was also capable of being modeled in three dimensional, visual/Euclidean space. But in order to do this, a *new meaning/interpretation* had to be given to the word “straight line”, namely, that a “straight line” was a great circle of a sphere in Euclidean space. In doing this Beltrami demonstrated that the formal logical consistency of this non-Euclidean geometry was on par with that of Euclidean geometry. He performed similar demonstrations by conceiving of other suitable interpretations for other non-Euclidean geometrical systems, like hyperbolic geometry. But, as was mentioned, the *meaning* of the term “straight line” had to be changed. It was changed from the notion derived from our concepts of “straight lines” in phenomenal visual space, to the notion of a great circle in visual space. Obviously, a great circle is not “straight” in the normal sense of the word (the sense of the word which Euclid’s geometry adapts to). As we know, great circles are curved, not straight. Anyone who views such visual models of non-Euclidean geometry will immediately recognize that the “straight lines” are not straight at all! As Morris Kline points out “To many mathematicians... these [Beltrami] consistency proofs were welcomed [because they] gave *meaning* to the non-euclidean geometries- *but only as models within Euclidean geometry*. Hence, one could accept them in this sense, but need not as geometries that might apply to the physical world with *the usual meaning* of straight line.”(Emphases mine)

As referenced, some people accepted the potential meaningfulness of the non-Euclidean formal systems given a reinterpretation of the word “straight line”- while rejecting that the systems could possibly adhere to the usual notion of “straight lines” derived from visual phenomena. As has been elaborated above, this position is warranted and correct. However, as Kline also points out, some were hesitant to accept the idea that the non-Euclidean systems could have any meaningful application to the hypothesized physical world. This is where the misstep occurred. For, even if we admit that Euclidean geometry is inherent to such notions as “straight line” and “space” as they are derived from our visual phenomenal experience, we must recognize that it is an *assumption* that these notions correspond to the characteristics of a hypothesized reality outside our experiences. One must give up the usual notion of “ straight line” if you are to understand the idea of non-Euclidean physical space.

The notion of “visually straight line” is phenomenally manifest to us in a number of different ways: The path of an object which no forces acting upon it; the path of a ray of light; the line/path of a stretched rubber band; the line/path of a chain with the least number of links from one point to another; the line/path of minimal mass of an object of uniform composition.[33] All of these phenomenal situations are perceived by us as "straight" in the usual (visual) sense of the word.[34] But, if we adopt the view that our phenomenal experiences are only indicators of things occurring in a physical reality outside our experience, then all of these referenced kinds of "straight" things, would be considered as corresponding to physical processes in reality. Thus, though the relations of all of these "straight things" may *appear* to correspond to our notion of "straightness" (and, thus, Euclidean relations) on the scales of our experiences of them, it is an assumption that these things *actually physically* correspond to such notions and relations. It is an assumptions that, on different scales of observation, we might not find that all of these processes appear to be "bent" or "curved", and/or that their measured physical relations do not adhere to those implied by Euclidean geometry.

For example, let us say that someone, perhaps named David, recruits three of his friends to perform some experiments in order to test out whether these things are always "straight" or not. David and his three friends put on space suits so as to be able to survive in the desolation of empty space, and, equipped with luminescent balls, high powered lasers, super stretchy rubber bands, long chains, and steel cables, proceed to travel to a portion of the cosmos remote from any other thing. David asks his three friends to assemble themselves in a very large equilateral-triangular formation from each other, while he flies far away from them to position himself so as to survey the triangle they form. Bright beacons are put on each of his friends so that he can see them from the vast distance. When David is sufficiently far from his friends, he asks them, via radio, to throw their luminescent balls to each other so that he can observe their path. David, with the aid of a cartesian plane drawn on the visor of his space helmet, observes the balls in his phenomenal field of vision as moving in lines which are *not straight* but *curved*! The path of the balls in David's field of vision are curved such that they seem to move outward from the path of a straight sided triangle (which he also drew of his visor for reference).



Instead of perceiving that the path of the balls are straight, as seen in the left triangle, David perceives that the balls move curved lines as seen in the triangle in the right.

[33] That is, the path that a bendable object which is of uniform weight over every equal portion of its length, (like a typical steel cable for example), which renders the minimum mass of the whole length lying between two points.
[34] Ignoring the physical assumptions involved (like possible forces -or lack thereof, as in relativity- etc.)

David, is perplexed and demands that his friends throw the balls to each other again, emphasizing that they not play any tricks on him. They do so, but, again, the balls move in the strange curves from each friend to the other, along paths which create a bloated triangle that looks as if it ate too much for dinner. David asks his friends to throw the silly balls away and instead shine the lasers on each other. They do so, but, yet again, David finds that the paths of the lasers is the same as the balls- they appear curved![35] David demands that the lasers be thrown away, and requests that the three friends stretch the super-stretchy rubber bands around them. Again, the lines of the bands between them appear curved. David has his friends try the same experiment with the taut rope, but the same result is registered. David comes up with a final scheme which he fancies will finally relieve his anxiety. He demands that his friends place lengths of steel cables between them. Naturally, the three friends pull the cables taut, and, as before, David perceives them to be curved in his visual space. He then demands that they place another set of cables between them, but place them in such a way as directed by him. Thus, one at a time, David directs each friend to place the steel cable along the path to the next companion such that David observes the path as straight in his visual field. After this is done two triangles are formed which share vertices with each other (with each friend at each vertex). One triangle appears to David as fattened by curved lines. The other triangle appears to be made up of straight lines. David then asks that the chords for both triangles be gathered and brought back to the space ship for mass-testing. Back on board, David and his friends find that the steel cords which appeared straight and shorter actually measured as longer and weighing much more than the other chords which looked longer and curved to David. David sits back in a chair and sighs: “But how am I to make anything exist in a straight line in space if even all of these things, which always seemed to be straight, now bend in space like silly gymnasts?!” One of David’s friends replies: “Well, what do you mean by space? Do you mean the phenomenal space of your visual experience, or do you mean the space hypothesized as existing in physical reality?” “What do *you* mean?” David replied tersely. His friend responds: “Well, if you assume that all these things we just experimented with will always adhere to ‘straight lines’ in physical space, then it doesn't really matter how they *appear* to behave in your perceptual visual space. For these physical processes are the only things which can give us an indication of the relations of the real lines of real space- you don't have any other way of finding out what the lines in real space might be doing. As for the Euclidean character of your phenomenal space, talk to Kant about how that is pretty much hard wired into your capacities for perception.” David thinks about it for a little bit, and comes to terms with this necessary distinction.

It is pointed out that the idea of “curvature” of space seems to imply that physical space must be curved with respect to something else. Indeed, there is no conceivable thing in physical reality which space could be curved relative to (since space is considered as the absolute which

[35] (Just pretend, for a moment, that you could see the laser path in empty space.)

everything else exists in and relates to), and, the methods of characterizing the metrical relations of space (including if those relations which embody "curvature"- or a deviation from Euclidean relations) does not depend upon reference to anything else. However, as the preceding short story of David and friends shows, it might very well be said that physical space is curved relative to the phenomenal space of our visual domain. There is a projective relationship between the two (if we assume physical space exists). If the projective relation is such that everything hypothesized to adhere to the "straight lines" of physical space actually appears straight in our phenomenal visual field, then we might say that real space is not curved relative to our phenomenal space. If not, we might usefully say that it is curved relative to it, in addition to physically deviating from Euclidean metrical relations.

Role of Geometrical Conceptions in the Generation of Scientific Hypotheses of Physical Principle

I would like to address one more issue respecting the concept of space with its corresponding logical geometries and their role in scientific thought. The issue is made clear by noting the observation made by people such as Poincare that the Euclidean and the non-Euclidean geometrical systems are both equally applicable to a characterization of physical processes. For example, thinking back to David and his friends, we can imagine that if the three friends were to pull three chains of identical length taut between them, David would perceive that the chains curved outwards. David would not necessarily need to assume that space is non-Euclidean however, for he could assume that the physical size of the individual links in the chain became longer and curved when pulled taught for some reason. After, all, this is what occurs in his *phenomenal* perception of the situation- why not assume that this also occurs in the physical reality of the situation? As another example which Einstein used in his lecture "Geometry and Experience": Euclidean geometry implies certain possibilities for the close packing of spheres in space. Non-Euclidean geometries imply different close packing of spheres in space. If our friend David were to try close-packing some kind of spherical thing, like large marbles of equal size, in space, he might find that the he is unable to pack them in the way that Euclidean geometry indicates he should be able to pack them. However, instead of assuming that physical space is non-Euclidean, David could simply assume that his marbles simply changed their sizes relative to each other as he was packing. He might even be able to find a certain point of view of his packed marbles which allows him to see the discrepancies in visual magnitude relations of some of the marbles from what they should be if they were of equal in size. David might, then, just as well continue thinking of Euclidean geometry as the real space of the world. He would, however, need to add a voluminous array of special rules and conjectures about how the dimensions of objects will irregularly warp in their size as they move through space. He would need to come up with more notions explaining -or at least characterizing- the bending of light and the motion of objects under no apparent force. David's scientific position would be much like that of Ptolemy as described by Benjamin Franklin: “Ptolemy is compared to a whimsical cook, who, instead of turning the meat in roasting, should fix that, and contrive to have his whole fire, kitchen and all whirling continually around it”. In David’s Euclidean worldview, light, the size of objects, new forces and masses would need to “whirl around” in a gargantuan convolution in order to remain consistent with the notions of Euclidean space, given manifestations of phenomena whose relations are more simply understood with non-Euclidean geometry. Thus, from the standpoint of actual scientific practice, the adoption of non-Euclidean geometry is better by virtue of the mere fact that it simplifies the way in which we are required to think about the world. Riemann also points of this aspect of the scientific import of the non-Euclidean geometries: “We are therefore quite at liberty to suppose that the metric relations of space in the infinitely small do not conform to the hypotheses of [Euclidean] geometry; *and we ought in fact to suppose it, if we can thereby obtain a simpler explanation of phenomena*.” (Emphasis mine)

Beyond, the benefits of conceptual simplicity and economy in the face of seemingly effectively equivalent hypotheses, there is another important point to consider, namely, the increased ability to predict never before seen phenomena with a new hypothesis. This applies to our choice of hypothesis respecting the metrical relations of physical space, and thus, Poincare was incorrect in claiming that theoretical adherence to one geometry is scientifically equivalent to that of another. For example: From the standpoint of an explanation of the motion of the planets, Einstein's theory of curved spacetime and Newton's theory of Euclidean space with a gravitational force are equivalent.[36] (Poincare might have pointed to this as an example of the equivalence of the applicability of different geometries to physical space.) However, Einstein’s hypothesis of a curvature of the physical space near a mass enabled the prediction of the bending of light around the sun. On the other hand, Newton’s Euclidean based theory did not enable this prediction. Thus, two hypotheses may be equivalent in their ability to embrace all of the currently known observed phenomena. But, one of the things that separates a *truthful* hypothesis from the rest of those sharing this equivalence, is the ability of that hypothesis to predict *new phenomena never before observed*. True, after such new phenomena is observed, new hypotheses, massively contorted and contrived, can be conjured up which provide equal logical derivability of the phenomena; and such could be the response of the “conventionalist” to this example. But the *truthful* hypothesis is always ahead, and the dead hypotheses are always playing “catch-up” in this respect.

Beyond this, there is another consideration of importance to address. Science is not an accumulation of descriptive relations subject to purely formal representation- despite the virulent claims, by some, to the contrary. Science deals with the use of distinct concepts, sometimes called physical principles, which enable us to grasp reality, conceptually and physically, in a way

[36] Besides the motion of mercury, which, however, can be explained away in the Newtonian system by hypothesizing another planet, or some other contrivance.

that increases our power to exist in this universe. There is, however, a relationship between the *descriptive* notions which we adhere to about the relations of things in reality, and the *causal* concepts which we believe to be the higher reason as to why those merely descriptive characteristics are necessary. The way in which we consider the mathematical relations of things will play a role in the kinds of physical principles we will conceive of as being the basis of those relations.

For example, take the history of Astronomy, as that history can be thought of as culminating in the discovery of universal gravitation by Newton: Up to the time of Kepler,the geometrical models created for the prediction of the positions of the planets were were thought of in a somewhat inconsistent way, perhaps even by their own creators. For instance, sometimes statements were issued from the likes of Brahe, Ptolemy and Copernicus, which indicated that they thought of their models as merely predictive tools, intended to be used to accomplish the great feat of predicting where a tiny dot in the sky called a star would be.[37] But, we also see statements from among them that they thought that the planets must actually move in circles in space for theological and philosophical reasons- which would seem to indicate that the models were thought of as accurate descriptions of the relations of the planets in reality. This point is made because, if mathematical models are made only for the purpose of correspondence to known observations and/or prediction, and not for the purpose of accurately describing the actual relations of things in reality, then no additional causes can be hypothesized as to *why* the mathematical model being used works for prediction. Further, if the mathematical model is taken as a description of the actual relations in reality, the question still remains as to *why* the relations of things in reality would be of that mathematical form. By posing *this* question, we establish a basis for new hypotheses respecting why the relations of those real things should be those found in our (assumed to be) descriptively accurate model and not otherwise. This is, generally, the point at which the scientist will resort to concepts at the level of physical principle- the intuitive notions which separate physics from mathematics.

Kepler was clear that his model for the relations of the planets was to be understood as an accurate description of the relations of the planets in reality. Thus, Kepler's mathematical model was capable of being subjected to explanation by physical hypothesis, and his model *suggested* such hypotheses.[38] Indeed, Kepler conceived of a dual-force physical hypothesis to explain the relations he found in the motions of the planets- one force originating in the sun which moved the planets in a circular path, and another magnetic force which moved them closer to and further

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[37] Each of these men might have had a fully consistent position on the matter. I don't know either way.

[38] Although, of course, Kepler always had a certain hypothesis as to what *caused* the motions of the planets- namely, a power emanating from the sun. As can be seen from a study of his work, this general physical hypothesis informed him throughout all of his scientific labors- including in his search for the proper descriptive mathematical relations of the planetary system.

from the sun as they proceeded around their otherwise circular orbit. Despite that Kepler did not come to the perfected notion of universal gravitation himself, his work was the immediate basis by which that notion was accurately conceived by Newton. For the mathematical relations Kepler had found in the motions of the planets indicated a pattern the *acceleration* characteristic of *all* the planets. This mathematical relation, assumed to accurately describe the relations of the planet's' motion, suggested the proper conception of physical principle: A force shared between all bodies and quantifiable in its effects in a way consistent with Kepler's law. The hypothesis that a force adheres between all corporeal substances according to a definite mathematical law was thus established. The highly successful elaboration of this hypothesis over the course of time led to a widespread agreement that it constituted the greatest scientific revolution in history, a view held by some to this day.[39] Thus, as can be seen, the creation of one of the most important hypotheses of physical principle in the history of science was directly dependant upon 1.) Considering the (Kepler's) mathematical model of relations as accurate to reality. And 2.) The unique characteristics of that mathematical model which suggested the proper form of physical hypothesis to be generated. (For example, even if Copernicus' model of the mathematical relations of the planets was interpreted as descriptive of reality, there is nothing in that model to suggest any coherent physical principle which could explain or it or coherently relate to it.)

Indeed, the power of a certain description of reality -whether it be mathematical or not- to suggest to the mind a generative causal principle might be called the *aesthetic* characteristic of any scientific theory; the *aesthetic* characteristic of any mathematical theory. For my part, at least, I can say that I am of the conviction that the perception of an *aesthetic* quality in something is identical to a recognition, by the mind, of the imminence of a truthful discovery of something good on the part of the mind which perceives it.[40] As a further amplification of this point, consider the following: Ask yourself the question- what does the solar system *really* look like? What is the *true* view of the solar system we should use? The fact is that, there is no "true" way of looking at the solar system, except as qualified by the notion of the *aesthetic.* For, even today, an infinite number of different mathematical models can be constructed which accurately describe all the relative distances and the motions of the bodies in the solar system. (I am not just talking about equivalence of predictive value, as Kepler pointed out as adhering in the models of Brahe, Copernicus and Ptolemy- but the equivalence of the measurable physical distances between the bodies of the solar system) For example, we are accustomed to imagine the solar system as viewed with a "god's-eye view" with the sun at rest and the planet in motion. But, we could just as easily view the system, even while preserving the physical distances, as from a view

[39]Some of Newton's own writings regarding this hypothesis, and the epistemological basis of its creation, are equivocal. Take the discrepancies in the content of the various editions of Newton's principia as an example. The substitution of the word "phenomena" for the word "hypotheses" in the opening section is of particular -almost philosophically scandalous- interest. I intent to look more into this issue and furnish a (hopefully short) report of my findings.

[40] I have made the first steps in an experiment to support this philosophical notion as it pertains to the nature of scientific discovery. Hopefully I will find a chance to complete the experiment sometime soon.

which holds the Earth as stationary, or Mars as stationary, or even the earth's moon as stationary, or any of the bodies in the system as stationary. The paths of the bodies in the system under such alternatively chosen views would be quite irregular, chaotic, and singularly *unaesthetic*. None of these views would appeal to our sense of *beauty* and, thus, none of these views would suggest to us a truthful physical principle of operation. Kepler's "view" of the system from the sun, however, is aesthetic, and leads one to find the higher physical principles of the solar system. (And in more ways than one- as shown in his *New Astronomy*, and that as shown in his *Harmony of the World*). Thus, as Keats said, "Beauty is truth, truth beauty". Of any theory we must ask the question: Where does it lead you- conceptually, and physically?

Thus have I provided the reasons as to why I disagree with the opinion of Poincare, and other "conventionalists", that the various geometrical systems are of equivalent scientific usefulness. However, it might be noticed by the reader that the last point I have made is less clearly related to the issue than the first two. That is, one might ask: "Why are you saying that the discussion of the way in which certain mathematical models will more readily suggest physical principles to the scientific thinker is pertinent to the issue of the mathematical models of space? Surely, the right mathematical models of the motions of bodies helps us come to hypothesize the physical principles which would provide us a cause as to why the motions of those bodies exist in the way they do. But when we are talking about finding the right mathematical model for the metrical relations of space, we have never tried to think of anything which would be the *physical cause* of those relations (as we do for the relations of the motions of bodies). That is a strange idea- the idea that something *causes* the metrical relations of space. Shouldn't we simply try to find out what mathematical model is experimentally verifiable as the most accurate and simple one for the metrical relations of space and simply accept that as a characteristic of reality which we don't need to try to hypothesize any explanation for?" Indeed, no one has ever tried, or perhaps even thought of trying, to conceive of a causal physical principle which provides a basis for us to understand why the metrical relations of space are what they are and not otherwise. Even in Einstein's relativity, we find no such attempt to address the question: though the metrical relations of space are put into mathematical relation with the presence of mass, there is no attempted hypothesis at *why* the metrics of space bear this relation to mass. I currently hold the view that, indeed, there is something to discover here- a new principle which will provide us with the basis for comprehending the reasons as to why the metrical relations of space are what they are and not otherwise.[41] But, such a discovery, if it is possible, would require the same two prerequisites as were found in the case of gravitation: 1.) The proper mathematical theory of the

[41] See my paper *Leibniz, Newton, Gravitational Waves and the Second Triad* for more on this issue, as it is related to the issue of the debate between the "absolutist" and "relationalist" views of space -as exemplified by the Leibniz-Clarke debate- in the context of the rise of a non-Euclidean view of physical space today.
https://www.findingprometheus.com/single-post/2017/06/02/Leibniz-Newton-and-the-Third-Triad

metrical relations of space are taken as an accurate description of reality, and 2.) The theory is *suggestive* of, and adaptable to, the proper physical hypothesis.

For this reason, I feel that Poincare was mistaken, and that, indeed, the mathematical model we choose to use in our characterization of physical space is of great scientific import. However, I would also emphasize the absolute importance and necessity of a proper use of the view which Poincare takes with respect to scientific work generally. It can be of great benefit to seek out the ways in which multiple theories are actually of identical import with respect to the relevant characteristics of the phenomena or physical process under investigation. After all, we see just how far Einstein was able to go by identifying the relevant equivalence of gravitational and inertial processes- he unleashed one of the greatest scientific revolutions in history. Often, the sober identification of an equivalence in seemingly different concepts' relevant content is the basis to shatter the unnecessary divisions of theory which held back the progress of mankind's search for truth.

V. What are Numbers?

In the first sentence of the first chapter of a book titled "What is Mathematics", the book's author, renown mathematician Richard Courant, claims that "Number is the basis of modern mathematics." To show that this sentiment is general in the community of mathematicians, Courant goes on to point out that the famous mathematician Leopold Kronecker once claimed that "God created the natural numbers; everything else is man's handiwork." But what are numbers? Answering the question is not easy. Indeed, Morse Kline once pointed out that providing an explanation as to what a number is, is utterly "beyond the capacity" of most mathematics teachers. As pointed out earlier, there has been controversy as to what constitutes a number throughout the history of mathematical thought. Thus, given that number is such a crucial concept in mathematics, it is no wonder that problems of a fundamental nature have arisen in mathematics. A clarification as to what numbers are should, therefore, help us clarify the basis of the paradoxical problems at the foundation of mathematics and provide new avenues for their resolution. As indicated above, the rational modality of a thing and its corresponding logical system are determined by the concept which is under consideration. Thus, clarifying what exactly is the concept of number in modern mathematics will be of use in resolving problems associated with the formal logical systems attributed to the operation of numbers.

As a first step in this effort, I will discuss the concept of *collection*, or *set*, and the idea of establishing a system of names for the ordered series of abstract collections.

The Story of The Man With No Numbers: A Model of the Development of the concept of Number in Human History

Let the following illustrative narrative be entertained: Imagine that there were a man who had never been taught anything about numbers or counting. Further, imagine that this man had never met anyone who had ever been taught anything about numbers or counting. Then, let the question be asked: would this fellow have any concept of *collections* of discrete objects? As pointed out before, multiplicity is a fundamental aspect of human experience, and it is therefore difficult to imagine a person who did not have a concept of a collection of multiple things. Even on the level of basic perception, when the mind perceives multiple things in close proximity to each other, there is a tendency to comprehend the multiplicity as subsumed by a "group" -by a *one*- which each individual thing is a part of.[42] Beyond that, this man would most assuredly have a concept of *possessing* multiple things. The things which he conceived of as possessing would, of course, constitute a collection.[43] It should not be too difficult, then, to see how the concept of *collection* is prior to the concept of *number*. For, as this man has shown us, a person can have a distinct notion of a collection without ever being taught anything about number. Because the concept of *collection* is distinct to the mind, it can be given a name, like "collection", "set", "group", "assemblage" etc. Thus, let us imagine that the man in our story has a distinct concept of what a collection of discrete things is.

We see that such a man would also have noticed, at one point or another, that different collections of the same kind of thing can be perceived as unequal to each other. The difference between two unequal collections of the same kind of thing would be the difference of perceived magnitude.This might be the space that the collection occupies in the visual field, or the length of time the collection requires to be surveyed as with a sequence of sounds. The man comes to understand this, and further comes to see how it is that the difference in magnitude is only modifiable in a discrete fashion[44], namely, that the magnitude of a collection is modifiable by at least one member. The man thus comes to form distinct notions of distinguishable collections- distinguishable from each other on the basis of their perceived difference in magnitude as well as multiplicity. It is not difficult for us to imagine how this man would be able to see that the same kind of collection could be found as composed of any kind of discrete thing. For example, the man would see that he has multiple hands- that is, he has a collection of hands. He could not help

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[42] See Wolfgang Kohler's "Gestalt Psychology"

[43] The concept of possession is a continuity which extends over many discrete things and subsumes them into groups, or collections. It is a readily understood concept. Indeed, the concept of "possession" is perhaps one of the most primitive notions. Babies, who do not yet even attribute words to objects, demonstrate possessiveness of objects. Even animals demonstrate possessiveness. This is probably why the concept of possession is so commonly used in the teaching of arithmetic to young persons. It provides a quick basis for the comprehension of the concept of "collection". For example: "If I *have* two apples and then someone *gives* me 3 apples" etc. -

[44] Granted the members of the collection are indivisible.

but notice that he also has a collection of multiple feet. Further, he would notice that the kind of collection he has of feet is equal to the kind of collection he has of hands. No counting or use of numbers would be needed for the man to realize this. It is easy to imagine how the man would recognize this collection in other instances, for the man could perceive his pattern of perception in any instance of encountering a collection, and on that basis judge that pattern to be similar or different from the pattern of perception associated with the consideration of the collections of his hands and feet. For example, he would probably very quickly notice that the collection of wings on a bird is the same, in some way, as the collection common to his hands and feet.

It should be pointed out that this man, although knowing nothing of numbers or systems of counting, would still be able to clearly understand the idea of adding to or subtracting from collections. He could even understand the meaning of dividing and multiplying collections in distinct proportions.. Dividing a collection in half, for example, would have a distinct meaning- breaking it into equal parts- one for each hand (or foot), while doubling a collection in size would also have a distinct meaning. That is, all the basic “arithmetical” operations would be known to this man without having ever learned anything about numbers. Some of the collections being distinct to him, he could proceed to give them names.

However, so long as he continued to distinguish the collections on the basis of perception of their relative phenomenal magnitude alone, the capability to distinguish the different successive collections would become very difficult -and quickly impossible- as each collection to be distinguished became more multitudinous. This is because the comparative differences between the collections decrease as they become larger. For example, the difference in perceptible impression between a collection of two visual objects and a collection of three visual objects is very noticeable. However, the difference in perceptible impression between a collection of thirteen visual objects and a collection of fourteen visual objects is not very noticeable (especially when not presented in a familiar ordered pattern). The difference in perceptible impression between one hundred visual objects and one hundred and one visual objects is usually undetectable. Thus, while the distinction and naming of collections on the basis of their perceptual impression might be reliable for smaller collections, distinction and naming of collections on the basis of their perceptual impression is not reliable for larger collections. For example, let's assume that our man lived in a home in the countryside and possessed a collection of cows. Imagine, then, that the man temporarily traveled from home, and returned after some time. How would he know whether he still had the same collection of cows as when he left? Such would be an important question- the herd of cows might have been reduced by thievery, or increased by calf births. If the collection of cows was small before he left, perhaps being a mere collection of what we call “2”, then, all the man would need to do to settle this question upon his return would be to look at his collection of cows, for it would be very easy to tell whether the collection was the same as when he left. The man would not need to perform any counting or other procedure to determine the equality of the collection of cows upon his return with the collection of cows present at his departure, for the particular collection of "2" would be very distinct to this man. But, if the collection of cows possessed by the man were much larger, he would not be able to determine whether the collection was the same upon his return on the basis of his perceptual impression of it alone. He would need some non-perceptual way of ensuring that the collection of cows upon departure and arrival were the same.[45] Yet, he would still not need to learn how to count or use numbers. One way in which the man could establish a method of determining the question could be as follows: He carefully rounds up all of the cows into an enclosed pen one at a time. As he does this, he marks a piece of paper with dots- he marks the paper with a dot each time he puts a cow into the pen. After having rounded up all the cows into the pen while making his marks on paper in this way, the man leaves his farm, reckoning that, when he returns, he will use his securely kept paper of dots to reveal if the size of the herd had changed while he was away. His procedure is simple: upon returning, the man carefully lets each cow out of the pen one by one. As he does so, he marks an "X" over one dot for each cow that he releases. If, when he is finished releasing all the cows, all the dots have been crossed off, he knows that the size of collection of cows he possesses is equal to what it was when he left. If some dots are uncrossed after letting out all the cows, he reckons that he lost some; if more cows are in the pen than he has dots to mark off, he reckons that he has gained some cows.

With the invention of the man's system of measuring collections of cows, the man has given himself a standard, albeit a rough one, by which all other collections of any kind of thing can be compared with each other: the more dots on his paper marked off by "X"s correspondingly matched to the members of some collection, the greater the collection. As an improvement to the system, he finds that placing the dots in an ordered pattern on his paper -most preferably in a straight line like so: ····················· - enables him to get a better idea of the comparative relation of the sizes, or magnitudes, of the collections he examines. For now, he can mark off the dots in order from left to right (for example), which enables him to more clearly discern the relative magnitudes of the collections he is measuring. The man would find it preferable to keep a standard piece of paper, with a long line of dots on it, and to mark the dots in order of left to right as he registered each cow moving in and out of the pen.We still see no use of counting or of numbers.

At this point, the man, realizing that all the possible collections between one[46] and the largest collection on his paper (that corresponding to the amount of dots on his paper) are contained on his paper. He sees that each collection in the system can now be definitely named. Before, the man would have been wary of trying to attribute names to the larger collections, because the

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[45] We will investigate the philosophical/physical assumptions utilized in this procedure later.

[46] We assume that the man has a concept of one- as we discussed that the principle of oneness is fundamental to conscious experience.

larger collections were not perceptually distinct to him and there would be great uncertainty in claiming that one large collection definitely was or was not the same as another large collection. [47]

The man labels the ordered dots like so:

Figure 1.

(some unique name Z)	(some unique name K)	(some unique name Y)	(some unique name L)	(some unique name E)	(some unique name F)	(some unique name G)	(some unique name H)
●	●	●	●	●	●	●	●

The names he puts above each dot correspond to the collection of the dot immediately below with all the other dots to the left of it (but it could be to the right if he preferred).

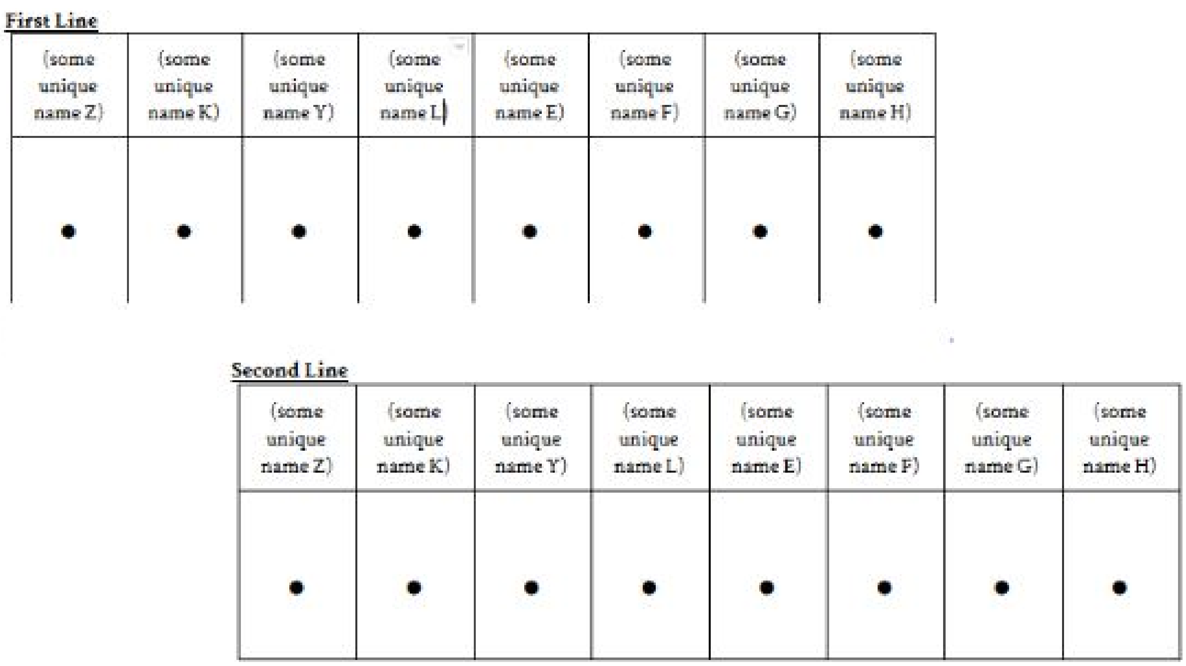
By labeling, or naming, all of the distinct collections in this way the man derives numerous benefits. For example, he can now memorize the sequence of the names for the collections in order of smallest to largest. By doing this the man no longer needs to go through the trouble of checking off all the marks on his paper for all of his cows every time he needs to leave home. For he can now *count* his cows- that is, utter the name of each collection in sequence of smallest to greatest for each individual cow noted. This saves him the trouble of dealing with dots on paper. If he properly counts all of the cows, he knows that, when he returns, he can count the cows again, and if the same name is reached, then the collection is the same (granted he remembers the proper sequence of collection-names). He can also go back to his paper and mark the exact dot in the line of dots on his paper which would have been the last one he marked had he marked the dots for each cow as he had done before. He might do this if he wanted a compact visual representation of the magnitude of his collection- something which he might compare with other such visual representations of other collections to enable a clearer discernment of their relative sizes.

As a further benefit of having named all the collections in his line of dots, he has the ability to precisely identify the collection which results from the modification of another collection, by addition, subtraction, multiplication, or division: He simply goes to his dots and finds the name of the collection which corresponds to the result of the modification of the collection in question. For this purpose, the man could draw, on a separate piece of paper, another line of dots and the corresponding collection names which is the identical to the standard line of dots on his first

[47] For the larger is the collection, the more similar are the collections which are most similar to it, and the smaller a collection, the less similar are the collections which are most similar to it.

paper (as in Fig 1.). If, for example, the man wanted to know what collection resulted from the addition of two named collections, say K and L, all he would need to do would be to place the second paper next to the first paper such that the dot lines were parallel and the first dot of the second line was placed next to the dot adjacent to that dot under the name K in the first line. Then, by identifying the name over the dot in the first line which was next to the dot in the second line under the name L, the name of the collection which results from the addition of K and L could be identified. Here is an illustration:

Figure 2.



In this case, the man would come to perceive that the addition of the collections of K and L resulted in the collection of "F". Proceeding this way, all of the possible arithmetical operations with collections can be performed, and the results tabularized in reference tables for future use. We see that, at this point, the man has provided himself with all of the basic tools by which systematic arithmetic is performed in mathematics today.

Continuing with the story of our hypothetical man, after he has proceeded through the previous steps, he notices that, although he has developed names for all the collections (up to the largest collection on his paper), and has tabularized all of the possible results of arithmetical operations with the collections (within the range of collections on his paper) there is still room for improvement in his system. For he sees that constant reference to the tables to locate the results of arithmetical operations with the collections is a cumbersome process. As just referenced, the man also finds that because he has given a unique, arbitrary, name to each collection only up to a certain point, there is no way to determine the names of those collections which are the result of the addition of collections he has named but which are larger than the largest collection he has named. Further, even if he does make up names for collections which are greater than any arithmetical result he will ever arrive at, the man will still need to empirically/physically perform the arithmetical operations on the physical collections (of lines of dots in this case) in order to discover what collection corresponds to the result of an operation- something extremely difficult, and actually impossible, for very large collections. Memorizing the entire sequence of names and proceeding to perform arithmetic by counting with something like his fingers would be even more difficult. Additionally, the man finds that it is difficult to call to mind the general size of each collection by the name attributed to it- that it is very difficult to locate a single named collection relative to the rest by its name alone- that there is no indication in the name of any given collection as to where it may stand in relative size to the rest.

The man realizes that all of these difficulties arise out of a lack of a *system* or *pattern* by which the collections are named. Upon realizing this, he has reached the final step which, once completed, give him the power to reckon with any conceivable collection. To restate: at this point, the man has established a system of identifying collections and their arithmetical combinations which is identical in all essential respects to the convention of collection naming (or “numbering” if you prefer) that we use today. The only difference between the two, as indicated, is the development of a pattern involved in the collection naming scheme which provides the basis for more convenient determination of the results of arithmetical operations upon collections.

The man realizes that any collection can be understood as the addition of smaller collections contained in it.[48] As a result, instead of giving a unique name to each collection, the man realizes that he only needs to name some of the collections and simply label, or code, all the other collections as aggregates of the collections he chooses to name. Obviously, this means that some collections will need to be given unique names which are not reducible to other collections (otherwise no collections could be indicated by reference to other collections). Further, the man needs to ensure that the collections which he chooses to name are capable of aggregating to form any possible collection. To ensure that all possible collections are reducible to the collections which he chooses to name, at least some of the named collections must immediately succeed or precede each other in the sequence of all collections ordered according to size. Without this, there would be some collections which were not reducible to the smaller collections named-

[48] Besides one.

there would be no way to fill in the gap between collections which are not next to each other in the scale of size.[49] He finds that in order to establish the capability for all collections to be named in terms of aggregates of smaller named collection, minimally the smallest two collections must be named.

Thus, the man selects a relatively small group of symbols to be the sole characters used in the naming and coding of the different collections: 1, 2, 3, 4, 5, 6, 7, 8, 9. The symbols, in the order indicated, are then used to name the smallest collections, ordered by size, like so:

1 2 3 4 5 6 7 8 9 ?

. .. .

....

In the case of the last collection in the sequence above, though the man might give it a verbal name of "ten", he has run out of symbols which he can give as a name/label to the collection. Thus he begins the process of symbolic coding. He creates a two digit symbol, the characters of which are the same as in the sequence above, excepting the use of "0" as an indication of "none". He attributes the following two-digit character to the final collection in the sequence above: 10.[50] This character is a code. The code is simple: the left digit indicates how many collections (in terms of the named smallest collections 1 through 9) of the last collection above ("ten") are in the collection indicated by the code; The right digit indicates the presence of a named smallest collection in the collection indicated by the code. Thus, The final collection in the sequence above is understood by the code 10 to be one collection of itself, and no collections of the named collections.[51] Thus the label, or code, "39" indicates the collection containing a collection of 3 collections of "ten" (or 10), and the collection of 9. The label, or code, "94" indicates the collection containing a collection of 9 collections of "ten", and the collection of 4. Etc.

In the case that the man tries to label a very large collection, he might find, again, that there are more members than he has symbols to indicate collections for. Any collection larger than the collection of 99 would present a problem for him. In order to make up for this, the man adds another digit to the left side of the code, and establishes that all characters placed in that position will indicate the collection (between 1 and 9) of the collection of a "hundred" (the name for the collection which comes after 99 in the collection size scale). "Hundred" is thus known as a

[49] In order to establish the capability for all collections to be named in terms of aggregates of smaller named collection (other than one- which would only be to name every collection uniquely), minimally, the collections of one and two must be named.
[50] It should be noted that the code system does not need to correspond to another set of verbal names that are attributed to the collections. For example, the code "10" has no unique characters in it, but the word "ten" is a unique word.
[51] To avoid confusion, the collection 10 is given a verbal name "ten", but, unlike the named collections up to 9, 10 does not have a unique name symbolically, being composed of the symbol for 1 and the placeholder 0.

collection of ten collections of ten. Thus the label, or code, of 233 translates into "the collection comprised of the collection of 2 collections of "hundred" (ten collections of ten), the collection of 3 collections of "ten", and the collection of 3".

The pattern described is established as his collection-code system, (or *number system* if you prefer). Thus, the man overcomes the need to name all of the possible collections individually, and, instead, names only a few collections, and labels all of the other collections as collections of those collections, or collections of collections of those collections, or of collections of collections of collections of those collections, (as this is represented by 10, 100, 1000 etc.), and so on.

The man then erases the old names for the collections which he originally wrote above the line of dots on his paper and proceeds to replace them with the pattern of symbolic names (and then codes) corresponding to his new system. After doing so, he begins to perform the operations of addition with his newly labeled dot system, and finds that the collection of 1 added to the collection of 1 results in the collection of 2; that the collection of 1 added to the collection of 2 results in the collection of 3; and so on for all the combinations of collections labeled 1 through 10. He records all the results in a reference table as before, but, this time he does not need to proceed any further in order to determine what the results of combinations of larger collections will be.[52] For all of the collections are labeled in the code he has established, and only with the symbols for the collections the total arithmetical combinations of which he has already exhausted and tabularized. Thus, the result of any arithmetic operation with collections can be found by performing the operation with the individual parts of the collections represented by code digits. For example, 84+77. The mind reads "a collection of 8 collections of ten, and a collection of 4, added to a collection of 7 collections of ten and a collection of 7" Knowing that adding the sub-collections of two collections will furnish the same result as adding the two collections themselves, he proceeds to add the sub-collections indicated by the code: The collection of 7 added to the collection of 4 yields a collection of 11 (per the table), which in turn is understood as 1 collection of ten and the collection of 1. The final step is adding a collection of 1 ten to the 8 tens to a collection of 7 tens. Since the collections are added in the same way no matter what their members are conceived to be, this can be done by simply adding the collections of 8, 7 and 1, which yields the collection of 16. But what are the members of this collection of 16? The members are collections of ten- there are 16 of them, which is 10 collections of 10 and 6 collections of 10. But 10 collections of 10 is the collection of 100. Therefore, the result is 1 collection of 100 and a collection of 6 collections of 10, and a collection of 1; or, in the code of

[52] It should be noted that here we are admitting an element of empirical investigation into the process of establishing a system of collections (mathematics). Another way to do it would be to algorithmically identify every possible combination of sub-collections which is in a larger collections, and tabularize the results. This process is the basis upon which all arithmetical operations could be interpreted. It seems that Russell, Hilbert and others were not satisfied by this kind of thing and demanded a way to deductively prove the thing which was already self-evident to intuition. It took Russel hundreds of pages to "prove" that 1+1=2 in his effort to do so- an effort which was proven futile by Godel.

the system, 161. This is the basis for the formal procedure known as “carrying the one” which children are trained to perform (usually without any understanding as to why it is correct) early in education.

As a quick note, it should be pointed out that the size of the collection of symbols used in the kind of collection naming system (number system) described will change the symbolic results of adding collections. For example, the system used in our story (and in modern mathematics) is a “base ten” number system, and this defines the symbolic results. For instance, 576+47=623 in a base 10 number system. But 576+47=645 in a base 8 number system. (The most basic collection naming system that can be conceived of is the base 2 system, or binary system, in which every collection can be coded with the use of only two symbols; usually 0 and 1. Today, the binary number system is the basis for coding mathematical statements into the electro-mechanical framework of computers.)

Thus, with the help of our story’s protagonist, we have identified the need to distinguish between abstract collections and the various names or symbolic codes which we may attribute to them.Systems of collection-labels can be crafted which provide easier determination of the results of arithmetical operation upon collections than non-systematically created number conventions. The formal rules governing the manipulations of the symbolic representations of collections must always have justification and foundation in the intellectually discerned relations of collections. In other words, the formal logical system which applies to numbers is a result of the rational modality corresponding to the notion of abstract collections. For example, the mind views the “five fundamental laws of arithmetic” as laws because of the way in which the mind thinks about collections. All of the “proofs” utilized in convincing people that the five laws of arithmetic are true make use of concrete examples of collections and their relations. Usually, the members of the collections used for the purposes of illustration are identical visual dots. In the mentioned book, Courant utilizes collections of visual dots to convince his reader of the truth of the five laws of arithmetic.[53]

Thus, if we are to understand what the word “*number*” means, we must be clear as to whether or not “number” refers to the *abstract collections* we have described, or whether it refers to the *names or symbolic codes for those abstract collections*.

Comments on Courant’s Explanation of Number

Though it may seem a basic point that the names or symbolic representation for something must be distinguished from the thing that is named or represented, it nonetheless seems that, because

[53] Except the fourth law, which he omits the illustrative proof for.

of the illustrated way in which the convention of symbolic representation is related to determining the results of operations with the collections, this distinction is sometimes missed. For example, take Richard Courant's explanation of "number" in the first pages of his referenced book.

Courant points out that, in modern mathematics, it is believed that all statements must be reducible to statements about natural numbers. He then goes on to attempt to explain where numbers come from, saying: "[Numbers are] created by the human mind to count the objects in various assemblages." Thus, at first, Courant seems to claim that numbers are *words* , created to count the members of assemblages, or collections- for it is inconceivable what else besides a word or symbol might be used to "*count the members of a collection"*. This indicates that Courant intends to say that the word "number" only refers to the labels associated, by convention, with the various abstract collections.[54] But, in the next sentence, Courant makes a confusing statement in trying to define a particular "number", saying: "The number six **is** an abstraction from all actual collections of **six** things." As can be seen, Courant has defined a word using the same word, namely "six"- which is quite unsatisfactory. Besides being an invalid definition, the statement is in contradiction with his first definition of what a number is: Something used to count objects in collections. Courant then claims "Only at a rather advanced stage of intellectual development does the abstract character of the idea of number become clear." But, again, Courant first claimed that the numbers are created by the mind to count the objects in collections, which would seem to indicate that he intended to mean that they are merely words used as labels for the members of collections. Yet, then, he claims that numbers are "abstracted" from experience. How are we to understand how something we create to refer to a thing can be abstracted from the experience of that very thing? This, again, seems to indicates Courant's confusion of the labels given to collections and the abstract notion of collections themselves. For him, the word "number" sometimes refers to the one, sometimes to the other. Continuing with his remarks, he claims that, "The mathematical theory of the natural numbers, or positive integers, is known as *arithmetic*. It is based on the fact that the addition and multiplication of integers are governed by certain laws." Here, again, we see the confusion manifesting itself. For if the numbers are only words used to count the members of collections, as he first claimed, then the idea of multiplying or adding *numbers* (which are words) has no meaning. Adding or multiplying *collections* however, does have distinct meaning. He goes on to say: "A concrete model for the abstract concept of integer (number) will indicate the intuitive basis on which the laws [of arithmetic] rest." He proceeds to provide illustrative proofs of the laws of arithmetic using visual examples of collections, namely, boxes with dots in them. Thus, he uses a model of collections to reveal the "laws of integers". But he claims integers are

[54] He then claims that "Numbers have no reference to the individual characteristics of the objects counted." But this cannot be the case, for the objects must all have the characteristic of objectivity, or discreteness, and they must all have the characteristic of membership in the collection labeled by the number used.

numbers. But he also implicitly claims that numbers are words used to count the members of collections. Therefore, if we apply the his first stated meaning of the word “number”, Courant claims, in effect: “We need a concrete example of collections to reveal the basis of the laws of the words invented to count collections.” It would have been much better simply to say that the basis of the laws of arithmetic, which are the laws of the combinations of collections, is illustrated by concrete examples of collections themselves. This is, in fact, what Courant proceeds to do, but, the term “number” is still shrouded in mystery because of the equivocal ways in which Courant used the word.

Here we thus see that there are two different kinds of “laws” which govern the formal treatment of arithmetical operations with collections. 1.) The laws which derive from the notion of the collections themselves. 2.) The laws which derive from the convention of symbolic coding by which the collections are labeled.

As far as I can tell, if we accept the abstract concept of *collection*, (along with the idea that collections can be either larger or greater than each other, and only modifiable in size by one member), and the idea of a *name* or *label* for each conceivable abstract collection, then there is no concept which the word “number” should correspond to which were not one of those two concepts.

So we must choose- do we define the word “number” as the *label* or *name* we give to the abstract collections represented by some symbol like “2”, or is number supposed to mean the *abstract collection itself*? Only by answering this question can we answer as to whether or not it makes any sense to say that symbols like $\sqrt{2}$ are symbols for *numbers* or simply an alternative *number* for labeling some collection. Because of this ambiguity, I will generally not use the word “number” but only “collections”.

Collection and the Fundamental Principles

Above, it was mentioned the concept of collection (then called number) seems to preclude any formal reduction of arithmetic. Is the notion of collection conceptually reducible to more basic notions? It might be construed that the concept of collection is a hybrid of fundamental notions like multiplicity and oneness. We might try to define a collection as “A one of an unchanging many ” or “A thing of many unchanging things”. But, whatever the case may be, this seems not to have any bearing on the question as to whether arithmetic can be reduced. In any case, the concept of collection is so primitive, that, if it not be identical with multiplicity itself, or even if it is conceptually reducible to some of the fundamental principles, it is a conception which is so immediately associated with the basic act of thinking that we probably don't need to worry about reducing it for any useful purpose.

VI. Collections, Magnitudes, and Proportions

The fundamental principles of cognition are the most basic concepts which can be obtained by the mind. Therefore, the attempt to reduce any subject of human inquiry to an elementary set of concepts which provide the basis for the formation of a logical system, cannot proceed beyond the fundamental principles. That is, the fundamental principles cannot be reduced to any other more elementary concepts- they are elementary and self-evident. However, even if it were true that the fundamental principles enumerated at the beginning of this paper were not the most basic of human mentation, the construction of logical systems on the basis of judgements respecting how these principles apply would still be justified. For these principles are of very clear meaning to the mind, and that is the requirement of any notion which is to anchor the rational consideration of any proposition. None of this should be offensive to the reductionist since, it must be admitted, that, at some point, reduction of concepts must cease at a set of elementary concepts, otherwise there would be no starting point to construct a logical system. In any logical system, certain concepts and axioms must be taken as self evident in their meaning and in their truth, else nothing could be deduced which were of any meaning.[55]

The fundamental principles of cognition are capable of being consciously reflected upon by the mind. Once the mind recognizes these fundamental principles, it is in a position to consider things with respect to them. For example, because the notion of "equality" is clear to the mind, we can judge things in our experience as being either in conformity with equality or not. The mind is also in a position to impose the principles upon experience, or, stated another way, entertain the hypothesis that things exist in absolute conformity with the fundamental principles. For example, the mind might wish to conceive that two things it finds in experience are equal, even though they may not really be equal. By artificially conceiving the things to be equal in this way, the mind, based on the meaning of the concept of "equal", can make inferences about the objects which would be necessary if the objects *were* equal. This process, utilized in mathematical physics, is commonly called "idealization". As another example, the notion of "one" is clear to the mind. Because we recognize that all things conceived of as one have parts, we are able to artificially construct a (one) thing out of any multiplicity of distinct things which we chose to conceive of as a parts of a one. Sometimes this corresponds with our perceptual predispositions, and sometimes not. For example, our minds have the tendency to perceive visual objects which are in close proximity to each other as parts of a "group". The study of Gestalt Psychology investigates such mental processes. But, the mind can also conceptualize different

[55] The question may be posed, if some concepts and axioms must be taken as self evident, why is it necessary to provide any logical justification for a proposition which is self-evident? What is the criteria for ceasing the process of reduction and finally accepting the authority of intuition (which is what happens any time someone admits something to be self-evident)

things to be parts of a "one" or "group" which are not disposed to perceptual identification as being subsumed in any "one" thing. For example, I can conceive of a one, or group, whose members are as follows: the notebook on my desk, the Eiffel Tower, the squirrel on the tree outside, and the Roman Empire. I have thus constructed a one, which includes multiple discrete parts- a collection, or set. But these things are not readily able to be *perceived* as being parts of one thing, as are, for example, the following *group* of dots: ········· The mind is what determines whether certain things are conceived of as part of a collection or not, and the determination is arbitrary, despite often being suggested by similarities of phenomena or social conventions.

Discrete Versus Continuous Entities

On the basic level, there are two kinds of "things" which can result in the mind's conceptualization of different phenomena or concepts as "one". One, which we have just described, is characteristically *discrete* . The other is characteristically *continuous*.[56] Both of these notions are capable of being abstracted from any experienced actual instance of them. For example, the concept of magnitude can be abstracted from the experiences of any actual magnitudes, whether they be visual, auditory, or kinesthetic etc. On the other hand, the notion of set, or collection, can be abstracted from the experiences of any actual collection of things, whether they be collections of visual things, auditory things, or kinesthetic things etc. [57]

[56] (This may seem to be a contradiction to what we said earlier, that any conceivable thing must have a continuous characteristic between its beginning and end. But, as noted, continuity may be conceived even when there is no perceptual continuity apparent. The non-perceptual continuity may be naturally evident, as in a piece of music, or it may be artificially imposed, as a self conscious notion, upon any things which we elect to consider as a part of a one or group. It may be said of a collection of arbitrary things, that the continuity is "My intention to count them all as part of a single group as I successively conceive of them.".)

[57] Because of the artificial character of some of the "ones", or "things", that the mind is able to conceptualize, the correspondence of these conceived things to any aspect of reality is questioned. It might be pointed out, for example, that just because different objects may be perceived in close proximity to each other and the mind might conceive them to be a parts of a "one", or group, this does not mean that there is a one distinct ontological thing in reality which these things are all parts of. There seems to be less of a tendency to object to the assumption that things which are perceived to be parts of a continuum are parts of the same thing. For example, if a person perceives a continuous extension of color in their visual field, they usually judge it to correspond to an aspect of reality which is also one continuous thing. But this is also an assumption. For there may be two distinct things which are not related at all which one mistakes for one, as is seen in optical illusions. But, in any case,as was demonstrated before, in any conceptualization of a thing, that thing must have parts. Thus, if we try to continually reduce all conceived things to their parts such that we avoid the idea of a "one over a many", as Aristotle would demand, we would continue on in infinite regression and never arrive at anything which we could take as the basic unit, or concept, from which all other things are built. As a consequence of this, it is necessarily true that the notions of "one" which subsume many things, must either correspond in some way to some aspects of reality, or, we must admit that the mind is not capable of conceiving of any aspect of reality. Indeed, some people are so hostile to the idea that the conceptions created in the human mind could have any correspondence to the reality believed to be outside our minds that they, like Hume, reject the idea of reality altogether, or, like Kant, take the "epistemologically agnostic" approach that reality is no meaning or accessibility to us whatsoever even if it does exist.

The comparative relations which these two kinds of things, or “ones” are capable of is dependant upon whether or not they are considered in the abstract or in the the specifically actual. In the case of magnitude the only ability to make comparisons of magnitudes is in the actual, or in the specifically abstract, but not in the generally abstract. For example, we can identify comparative relations of actual phenomenal visual magnitudes or of abstractions of experienced visual magnitudes as in geometry. But can not do so of the general notion of magnitude- geometry will not apply to the general abstract concept of magnitude. All systematic treatments of magnitude are specific to the kind of magnitude considered. For example, the relations of visual magnitudes found in geometry do not apply to the relations of the magnitudes of sound or touch .

Collections have different levels of abstraction as well. For example, the abstract notion of the unique collection we call “two” can be abstracted from all actual instances of collections which include actual things countable to “two”. The same can be done for all actual collections which include actual things counted to “three”. And so on. Yet, an even further abstracted notion of collection can be abstracted from the just cited cases of unique abstract collections.Thus, there are different levels of abstraction. On one level, we may conceive of the general notion of number, or collection; on another more specific level, we may conceive of a *particular* collection in the abstract; finally, we can conceive of an actual collection of actual things which corresponds to a specific abstract collection. The abstract notion of any specific abstract collection will have comparative relations with every other specific abstract collection. These comparative relations are revealed by considering the abstract collections in one to one correspondence with each other. This is the basis of the notion of “greater or lesser” which is specific to the abstract collections. This notion of “greater or lesser” is related to, but distinct from, the notion of greater or lesser which is found in the relations of continuous magnitude. The proof of this is as follows:
Because, as we proved earlier, the existence of any actual thing must be expressed in some relative magnitude, any actual collection of actual things will be associated with some actual relative magnitude. For example, in any actuall collection of visual objects, the visual objects will each have a relative visual magnitude. Thus, the collection of such visual objects can be said to have a certain actual visual magnitude which is the aggregate sum of the visual magnitude of each of its individual members. As another example, in any actual collection of individual sounds, the sounds will each have relative magnitudes (like volume or length in time). Thus, the collection of such sounds can be said to have a certain actual aggregate magnitude of volume or time (depending upon the way in which the sounds are heard- sequentially or simultaneously) which is the sum of the magnitudes of each of its individual members.

Now, imagine that we are confronted with an actual collection of visual objects corresponding to the abstract collection of “two”, or “2”. Then suppose that we are confronted with another actual collection of visual objects corresponding to the abstract collection “four”, or “4”. If we examine the comparative relations of the *abstract* collections of 2 and 4, then we find that the collection of 4 can be equally divided into collections of 2. Or, alternatively, we find that the collection of 2 can be put into one to one correspondence with the collection of 4 precisely 2 times. Does this necessarily mean that the *actual* collections of 2 and 4 visual objects which we are confronted with also share this relation in the comparison of their actual aggregate magnitudes? For example, will 4 visual objects necessarily occupy twice as much (two-dimensional) visual space as a collection of 2 visual objects? No. For the visual objects of the actual collection of 4 visual things might be much greater or lesser in magnitude than that of the visual objects in the actual collection of 2 visual things, and, as a consequence, the comparative relations of the aggregate magnitudes of these actual collections will differ from those of their abstract counterparts. For example: 1000 grains of sand compared to 2 beach-balls- There are two ways of comparing the magnitudes of these actual collections: The visual magnitude of the collection of beach balls is much greater than the visual magnitude of the grains of sand, yet, the abstract "collection-magnitude" of the collection of sand grains is much greater than the abstract collection-magnitude of the collection of beach-balls.Thus is it proved that the comparative relations found to exist between the different abstract collections do not necessarily correspond to the relations of the actual aggregate magnitude of the members of instances of those collections of actual things. [58]

However, it is possible to enable different actual collections of actual things to exhibit aggregate magnitudes which correspond in their comparative relations to the comparative relations of their counterpart abstract collections. That is, it is possible to make the comparative relations of the aggregate magnitude of actual collections of actual things correspond to the comparative relations of the abstract collections which those actual collections correspond to. The way in which this is accomplished is by considering the members of the different actual collections to be absolutely equal to each other in some actual characteristic. For examples: Different actual collections of equal lengths, like meter sticks, or different actual collections of distinct equal kilograms. In these cases, the comparative relations of the aggregate magnitudes of these actual collections will be exactly equal to the comparative relations found in the abstract collections which they correspond to. For example, the comparative relation between the spatial magnitudes of an actual collection of 2 yardsticks and another of 4 yardsticks will be exactly that of the comparative relation between the abstract collection of 2 and the abstract collection of 4 (the characteristics of which were mentioned above). Thus, if actual collections of actual things share a common member in this fashion, the comparative relations of their associated magnitudes can

[58] Further, if we chose to identify a collection of actual things which do not share a common quality, and thus whose magnitudes are not capable of being intelligibly added together or compared- we find that some actual collections may have no actual characteristics which are comparable or related to each other at all. For example, a collection of 3 smells, and a collection of 3 lines have no actual comparable magnitudes- they are only comparable abstractly, on the basis of the common abstract collection they correspond to.

be determined by systematic treatment utilizing the formal system derived from the consideration of abstract collections (the formal laws of arithmetic) and the convenience of the collection-code system. This is the reason that common/standard units are requisite to the application of formal mathematical logic to distinct actual subjects, as in science or geometry. Instances of collections in which the members are equal to each other is only a special case of collections. It is this special case, however, which renders useful the application of the logical systems and naming-system for the collections to the description of empirical phenomena and the calculations of physical-scientific investigation.

The Power of Systematizing the Names of the Abstract Collections

The task of fully elaborating the specific benefits to mankind which arise out of the innovation of the modern collection naming system is too large for this paper, or probably any paper. I will mention, however, the most basic advantages which this system offers. The problems leading to the innovation of the collection-naming system of the type in use today has been touched upon above. All of those referenced problems find solution in the development of the kind of collection-naming code we have today. Namely, the ability to determine the name/code for any collection which will result in any arithmetical operation with any collection. It may seem to us that this offers us little advantage since we are just giving out names, and simply giving names to things does not make them in any way more useful, accessible ,or understandable. But, it has been found that the formal systems pertaining to the operations of *abstract* collections can apply very well to *actual* collections of many different kinds of *actual* things.[59] Thus, we can find the name of the collection of actual things which results from various empirical/physical modifications of actual collections of actual things- (as long as the individuality of the actual things being modified is not lost). Because the collection naming scheme, in progressive succession from least to greatest, follows a well defined pattern, if the name of the collection of actual things we seek is known, then, empirically/physically[60], we have the ability to create that collection of actual things with as much exactitude as is possible. How is this done? All we need to do is to keep adding the individual units of the actual thing intended as the basic member of our desired actual collection as we *count* to the collection-name corresponding to the one we need. As mentioned, since the counting process is well defined we know that this procedure will result in the right collection.Thus, we have established empirical/physical standards in society to be the basic units which all other collections are composed from[61]; this is the basis for scientific

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[59] Although much more will be used on this later, I will note that this is particularly true of those actual things which can be idealized as individual and *one* despite changes in time and empirical/physical context.This is mainly true of actual things whose individuality remains distinguishable over many situations. If actual things are chosen which are not susceptible to this kind of idealization, then the logic of abstract collections will not be of much use for the investigation of them. This is why arithmetic does not apply in many actual instances, like chemistry for instance.
[60] The distinction of these terms will be elaborated more later.
[61] Except those whose members are smaller than the standard units. This will be discussed later.

measurement upon which all the advanced technological processes of modern society depend. We can check to see if a an actual collection created by a empirical/physical process deemed equivalent to arithmetical modification actually corresponds to the name that it should had those operations actually been performed. (Something quite important in exchanges of valuable goods or money). How is this done? All the person needs to do is to *count* the actual members of the actual collection in their possession. Most of the time, the collection of actual things under consideration is so large that specially designed automated systems perform the counting of actual collections of things- saving massive amounts of time, and also enabling certain collections to be counted which otherwise never could be empirically counted in the lifetime of even the longest living person. Automated counting systems also overcome the problems with empirically counting actual collections which arise out of the limitations of our memory- some collections are so large that their collection-codes/names contain many digits- more than our memory can handle if we are counting without some sort of aid.

In short, the systematization of collection-names allows for people to precisely communicate to each other either the exact -or the almost exact- amount of things which are of interest to them.

Proportion as a Kind of Magnitude

The relation of different magnitudes discerned by the mind can itself become an object of consideration. It is found that the relations between different magnitudes can be more or less *extreme*- that is, the closer a relation corresponds to equality, the less extreme, and the further the relation is removed from equality the more extreme it is. Thus, we have two kinds of magnitude. The first kind of magnitude is the phenomenal magnitude which expresses some kind of phenomenal quality like color or sound. The second kind of magnitude is the relation, or *proportion* between different magnitudes of the first kind, relative to other such proportions.

VII. The Use of Collection-Logic and the Origin of Problems of Number

The evolution of the ideas associated with the word "number" is bound-up with the process by which man has attempted to find ways of applying the benefits of the formal devices (like the laws of arithmetic and the collection-code system) associated with abstract collections to problems which confront him in everyday life, and in scientific investigation. Of particular importance in this is the attempt to apply the logic of abstract collections of distinct things/concepts which are *continuous*. Because the rational modality of concepts of *continuous* things is different than the rational modality of the concept of *discrete* collections of things, the attempt to apply the logic of discrete collections to things which are continuous always falls short in some way, unless the logic is modified to make up for the discrepancies. A common way in which the discrepancy between the applicability of the logic of collections (of discrete things) and the concepts of continuous magnitudes is dealt with is by the invocation of the concept of infinity. Some examples of this include the case of the *calculus*, as well as cases of certain kinds of "number". The following discussion should provide illustrations of this point.

Collection-Logic and Non-Collection Logic (or Numerical and Non-Numerical Logic)

Briefly, before proceeding, I wish to discuss the difference between collection logic and non-collection logic of what are normally considered mathematical statements- statements which make reference to collections (or numbers). The "collection logic" of any statement about things which makes reference to their collection characteristics (or numerical characteristics if you prefer) is the logic of abstract collections. The logic of abstract collections is based upon a consideration of collections of things whose characteristics are completely unconsidered, save that they are individual "things" considered as in a group capable of being counted by our collection-code sequence. The "non-collection logic" of any statement about things is dependant upon the all other characteristics of those things other than their characteristic of being individual things in a countable group. These characteristics are notions which may be empirical, physical, or conventional, and they will determine the logic associated with any statements about them. (As I indicated earlier, the formal logic associated with any notion is dependent upon the rational modality of the notion itself, and nothing else).

For example, we might say that 1 raindrop plus 1 raindrop equals 2 raindrops. This is true only if we interpret the words in the statement, like "plus" and "equals", to adhere to meanings intended for discussions of abstract collections- to mean, for instance, that we consider the raindrops as *individual* things and *choose* to consider them as parts of a collection. The collection logic tells us that 1 thing plus 1 other thing is considered as 2 things: we can count the raindrops that way. However, we might find that some people would object to the assertion that one raindrop plus another raindrop equals two raindrops because of the empirical/physical way in which multiple drops of water combine with each other to form larger drops. This may lead someone to claim that, really, “1 raindrop plus another raindrop will result in 1 raindrop.” This, of course, depends upon that interpretation. We could try to establish a new formal logical system which captures this empirical/physical relation of raindrops as separate from the collection logic. It would be very easy in fact: a+b=1 where a is any collection of raindrops and b is any collection of raindrops and “+” stands for physically combining them.

Let us take another example: debts and assets. The existence of debts and assets in human society is often referenced as a justification for the notion of “negative numbers”. That is, it is claimed that the formal system involving the symbols called “negative numbers” is useful for treating certain processes in human activity- as in commercial or economic matters. Let's see if this is a good argument for the “existence” of “negative numbers”. Remember, the collection logic of any statement will only refer to the interpretation of the meaning of the statement in the terms of collection logic: the subjects referenced in any statement are assumed to be capable of being considered as individually conceivable things in an abstract collection- the entities are considered to be countable in the same collection. So, if we have 1 debt of a penny and 2 assets, each of a penny, how many things do we have? We have 3 things. Since it does not matter what the things might be if we consider only collection logic, it does not matter that in real life, debts cancel out assets in a certain way. 1 of *anything* plus 2 of *anything* will equal 3 *things* according to collection logic. However, if we wish to establish a formal system which captures the relations of these actual things as we deal with them in real life, we can use the formal system which utilizes the “negative numbers” if properly applied- as, for example, with the collection of debts being represented by a “negative number”.

Again, even though 4 sticks of dynamite plus 4 lit matches will result in 1 big explosion, we do not say that 4 sticks of dynamite plus 4 sticks of matches equals 1 explosion if we only consider collection logic. By collection logic we simply count all of the things chosen to be counted as part of the same abstract collection, which, in this case would be 4 things plus 4 other things for a total of 8 things.

In the science of chemistry we find many formal logical devices which do not rely on collection logic for the reasons here the subjects which the chemist deals with do not interact with each other in a way which corresponds to collection logic. Collection logic is thus of no use to the chemist who needs to calculate the actual results of various chemical combinations.

As stated in the last section, the ability of the logic of abstract collections to apply to actual things and their actual relations depends upon whether the distinct individuality of the things under consideration is lost in the course of the empirical and/or physical changes they undergo. If their individuality is not lost during the process, then the process is capable of being adapted to the formal logic of collections if the right interpretation is adopted for the symbols like +, -, =, etc. That is, if the appropriate aspects of the process are attributed to the symbols like +, -, =, etc. Because the distinction between collection logic and non-collection logic is not clearly identified, there is a tendency to confuse the meanings of the words and/or symbols used in both kinds of statements. For example, the word "plus" can sometimes be of purely arithmetical meaning, while, at other times, it can have various empirical/physical meanings depending on the context and the subject/process.

The Different "Kinds of Number"

Considering the way in which each of the different concepts which all came to be labeled as "numbers" were developed from the standpoint of the two types of ways in which logical systems are generated, will assist in the clarification of the issues attending them. With that, below is my thinking respecting the various kinds of "number".

-*"Natural Numbers" and "Whole Numbers"*

As I have mentioned, it seems to me that the only concept which should correspond to the word "number" is that of abstract collections. Thus, the "numbers" classified as "whole" and "natural" are the only set of symbols which corresponds to all of the abstract collections. Although I may dispute the inclusion of 0 as an abstract collection, since, after all, the symbol "0" only stands for the concept of "nothing". And since abstract collections are meaningful things, it seems that we should not include the concept of "nothing" in the same thing-category as those collections.

In short, I would say that, if you can't count to it, it isn't a collection (or number if you prefer).

-*Integers*

As mentioned above, if we take the word "number" to indicate the idea of *abstract collections*, then the term "negative number" have no meaning which is of the same nature as that of "numbers"- though other meanings, corresponding to non-collection logic can be attributed to that term. In short, there are no such thing as "negative numbers", at least as most people would *try* to interpret that term. My arguments for this are presented above.

-*Real Numbers and Rational Numbers*

--*Decimals*

The decimal numbers are a clear example of the way in which the conceptual -and therefore logical- discrepancy between the notion of collections of discrete things and the notion continuous magnitudes led to the creation of new concepts and formal devices purposed to narrow that logical discrepancy. (In this case, I would note that that logical discrepancy was narrowed to an indefinitely small, but not insurmountable amount.)

The formal system of the collections only provides advantages in real life if the actual subjects under consideration with the aid of collection logic are equal to each other and conceivable as discrete individual things. Therefore, we find the use of collection logic especially suited to aid us in consideration of actual things which are considered to be *indivisible* things of a common type. Despite the great advantages that collection logic offers for things of this sort, we find that collection logic fails us in cases in which the subjects under consideration are *divisible*. This is the case for all continuous magnitudes. As mentioned above, an arbitrary standard unit must be established for all cases of actual continuous magnitudes in order for collection logic to apply to them. But what if we are interested in amounts of actual continuous things which are not equal to the standard unit or its multiples? For example, a cloth merchant charges a certain price per meter of cloth, and uses collection logic to calculate exactly what to charge any customer who might buy any collection multiple (numerical multiple) of meters of his cloth. But what if a customer visits the merchant's store desiring to purchase only a portion of a meter of cloth, or perhaps more than a meter, but less than 2 meters? What if the customer wants to purchase a multiple of a certain length of cloth which is unequal to the merchant's standard meter-unit? The solution to this kind of problem is the decimal "numbers". But the "decimal numbers" are not really "numbers" (collections) as shall be shown. For a decimal number is only a regular collection with a proportion tacked onto the side to allow for any arbitrary continuous magnitude to be numerically related to any other arbitrary continuous magnitude (to any desired degree of precision).

For example: 2.5 is only shorthand for "The collection of 2 plus the amount resulting from the application of the *proportion* of 5/10 to whatever is being counted". Take this: 2.7(of a thing) = 2(of the thing) + (.7 of the thing). Since .7 only means "the proportion of 7 to 10 applied to the thing", or "7 pieces of tenth parts of the thing", the symbol 2.7 means "2 (of the thing) + (7 pieces of the tenth parts of the thing). From this simple concept of what a "decimal number" is, the formal operations with them are intuitively clear.

For example. 3.4 and 5.2. We find adding to be simple: 3 of anything plus 5 of the same thing will result in 8 of the thing. 4 of anything plus 2 of the same thing will result in 6 of the thing (which in this case in tenth parts of the thing). 6 pieces of tenth parts is represented as .06. So we have 8+0.6 or 8.6. How do we multiply them? Since 3.4 is shorthand for 3 + (4/10) we know that the answer can be found by adding the result of 3(5.2) to the result of (4/10)(5.2). This can be done as follows: 3(5.2) = 3 (5 + (2/10)) = 15 + 6/10. 4/10(5.2) = (4/10) (5 + (2/10)) = (20/10) + (8/100) = 2 + (8/100). (15 + 6/10) + (2 + 8/100) = 17 +(6/10) + (8/100), or 17.68. [62]

--*Rationals*

Comparing one collection to another in either abstract or actual magnitude is a simple matter. However, we may want to know more than merely that two collections are of unequal size. We may ask *how much* larger is one collection than another. This question presupposes some meaningful way in which the relative difference *of the difference* between the size of the collections can be indicated. This is provided by the concept of collections themselves. For the clearest way to indicate the difference in size -the clearest way to indicate the *comparison* or *proposition*- is by identifying how many times the smaller collection can be "put into" the larger. Or, alternatively, how many of the smaller collections exist in the larger. Thus, when someone says that one thing is 3 times larger that the other thing, we have a clear idea of what is said. But, are we referring to a *collection* (or *number*), or are we referring to a *proportion*. Although we may indicate the *proportion* existing between two numbers or magnitudes with a collection name, this does not mean that the *proportion* is itself a collection (or number). When we say the proportion of 9to3, or 9/3 is 3, we have a clear idea of what is meant: 9 is larger such that 3 goes into it 3 times. Thus, it seems that "rational numbers" are not numbers at all, but, rather, proportions between given numbers which the name of a number can indicate by telling us how many times the smaller "goes into" the larger. In response to this, a fellow may say that "Well, we simply *define* "rational numbers" as those numbers which result from the division of one number by another." This is well and good, but, most of the possible combinations by which the collections (numbers) can be divided by each other do not render exact collections- but, rather, render results of the form: "The collection of x plus a collection of y members". For example, 11/4= "2 with a remainder of 3". Thus, we would need to conclude that not all compared collections (numbers) provide us with a number, or collection, to indicate their proportion- and therefore, we cannot say that all divisions result in "rational numbers". In response to this, the fellow may say: "Well we can use decimal numbers". The decimal numbers do indeed patch up more of the holes in the array of missing numbers intended to express the proportion of different collections. For example, we did not have a collection (number) which could be given to indicate the proportion 11/4; but, a collection with a proportion tacked on (or a "decimal number" if you prefer) can indicate it, namely 2.75. But, another problem arises: some combinations of collections, when divided by each other, do not even render decimal numbers of any finite

[62] It should be pointed out that multiplying two proportions like 2/10 and 4/10 requires a different idea of multiplication. Here we are no longer substituting a collection of members for each member of another collection, as we thought of multiplication of collections originally. Rather, we are *applying the proportion* of the value of the thing relative to 1 to the other value relative to 1. Multiplication of collections can also be reinterpreted this way.

expression. Some instances, like 10/3 seem to require an infinite progression of decimals (3.333…etc) to be able to be understood as capable of being multiplied by the denominator to make the numerator (else, if the sequence of 3's terminated at some point, the result would fall short of 10).Thus, it would seem that even with the use of "decimal numbers" we cannot represent the proportions between collections by a single finitely expressed value. However, if we admit such infinite decimal "tack-ons" to our decimal numbers, then we can claim that we have filled all the holes in the attempt to providing "rational numbers" to indicate the proportions of the basic collections. Thus, according to my view, only some of the "rational numbers" are actually numbers (collections), most of them are collections with definite proportions tacked on (decimal numbers) which themselves are either finitely or infinitely expressed.

--*Irrationals*

It was eventually found that there are some geometrical magnitudes in certain constructions which can not be related by numerical proportion. These were, and are, called incommensurable magnitudes. Examples include the diagonal of a square to its side, and the circumference of a circle to its diameter. Algorithmic procedures were found which enabled the proportion of these magnitudes to be expressed in an increasingly accurate degree, while always falling short of definite numerical precision. Symbols were provided to act as representatives of these proportions, like π . But, this did not stop people from claiming that these proportions were also "numbers". But why would it be asserted that these proportions were numbers? Even the proportions which are defined numerically (rational numbers) are not really numbers, but proportions indicated by numbers (collections) which tell the multiple of the denominator needed to furnish the numerator- and that requiring artifices such as decimal numbers and infinite series of decimals. Sometimes, it is argued that the "irrational numbers" must be "real numbers", or "actual numbers", on the basis of their usefulness for calculations. But we should remember that symbols in a formal system can be used as if they represented actual numerical values even if they do not. If, after performing formal symbolic manipulations with such symbols (like π), we obtain the symbolic arrangement conducive to our desired calculation of an actual numerical value, the symbols for irrationals must always be replaced by a decimal number approximation if they are to be of any actually practical use. Thus, the symbols act like placeholders in the formal system for a *potential* number (as opposed to an actual number). Again, we point out that π is indeed a real proportion -at least as real as any conceivable proportion of any abstract continuous magnitudes-; it is the proportion of the geometric magnitudes of the circumference and diameter of a circle. This is the *meaning* of the symbol π.But the concept of proportion and the concept of collection, or number, are different. The inability to encompass the proportion value of π in the language/system of collections, or collections with decimals, is another example of the formal problems inherent in the attempt to apply the logic of the discrete to that of the continuous. However, practically, it makes no difference at all as I will explain.

Given the fact that no matter how small the equal divisions of a continuous magnitude might be conceived the parts of the magnitude so divided will always have some amount of actual length, it should not have been such a surprise to the world to discover incommensurable continuous magnitudes- such as the side and diagonal of a square. But, it should be noted that, although some continuous magnitudes can be shown to be insusceptible to precise numerical relation, this does not mean that these magnitudes do not have comparative relations with each other- this does not mean that they are *incommensurable per se*. For the diagonal in a square is easily compared to the side of the square and judged to be larger than the side, for they are both entities of the same quality and kind- they are both lines. This point is only made as a reminder that numerical incommensurability is not conceptual incommensurability. Numerical incommensurability of continuous magnitudes arises out of the conceptual and logical discrepancies between the continuous and the discrete. However, these discrepancies can be made infinitely small with the aid of devices like decimal "numbers". In fact, it should be pointed out that the numerical incommensurability of continuous magnitudes, as found in geometry, has absolutely no significance or bearing on anything of practical import for mankind whatsoever. For even if we admit that there are continuous magnitudes in reality[63], as well as numerically incommensurable magnitudes, then it would make no difference at all to mankind. For, anyway, we are never able to know the precise relative dimensions of any actual magnitude to any indefinitely small degree. Our ability to know the relative dimensions of a magnitude is dependant upon the precision of our observational capabilities- which are always limited. Since our observational capabilities will always be limited, and never absolutely exact, we necessarily admit that the "problem" of numerically incommensurable magnitudes has no significance for us whatsoever, since, by use of the decimal system, we can provide a numerical approximation of any two magnitudes' proportion to any degree of accuracy- including to degrees of accuracy beyond anything we are capable of -or ever will be capable of- verifying observationally/experimentally.

Imaginary and Complex

Having already discussed the case of " $\sqrt{-1}$ " and my opinion as to its meaning, I will briefly point out that Gauss' interpretation of this symbol as included in a special kind of graphical

[63] We may hypothesize otherwise of course. Some people in history fancied that there are little identical atomic spheres which make up all things. If such a thing were true, then no numerically incommensurate magnitudes would exist in reality- since all things would be composed of discrete identical building blocks (and therefore all things would have numerical proportions). Conversely, if this atomic theory were true, no magnitudes found to be numerically incommensurable in the abstract, like those found in geometry, would actually be physically constructible.

procedure/function is just as easily put into an alternative formal symbolism. For example, given a cartesian plane, we define all the units of length to the right of the origin on the x axis as “Right numbers” or (R), all the units of length to the left of the origin on the x axis as “Left numbers” or (L). All the units of length on the y axis above the origin as “Up Numbers” (U). And all units of length on the y axis below the origin as “Down numbers” or (D). Any point on the plane can thus be defined by indicating the collection of each direction to add units from the origin. This can take the form of (zR, yL, xU, wD), with the coefficients of the “direction numbers” being some number.

Now take the following rules:

$R^2 = R$	$L^2 = R$	$D^2 = L$	$RL = L$
$RU = U$	$LU = D$	$UD = R$	$U^2 = L$
$LD = U$	$RD = D$		

These rules furnish the same results as Gauss’ complex plane interpretation of the negative numbers and $\sqrt{-1}$. The points on the plane are each given a coordinate “number” of the form (xN + yM) where x and y are normal numbers and N and M are either R, L, D, or U. These coordinate values are then “multiplied” in the fashion Gauss multiplied his “complex numbers”. The result is a precise value of how many units up, down, left, or right one must proceed from the origin to reach the point defining the end-point of a newly created magnitude originating at the origin. Try it for yourself to see how you like it if you wish.

Thus, a graphical scheme has been created which is of precisely the same use and characteristics of Gauss's complex plane system. Although this one does not use $\sqrt{-1}$ .The point is that, the mysterious symbol of $\sqrt{-1}$ never really had to be used to create the system Gauss created. The attempt to reference Gauss's system as a reason why $\sqrt{-1}$ should be considered a “real number” is not very compelling for that reason.

Hypercomplex Numbers

“*The introduction of quaternions was another shock to mathematicians. Here was a physically useful algebra which failed to possess a fundamental property of all real and complex numbers, namely, that* $ab = ba$” -Morris Kline

The development of the quaternions and their meaning could be said to be a “Type 1 system generation, while the development of the imaginary numbers and their meaning could be said to be a “Type 2” generation.

The formal system of quaternions was crafted to a very specific purpose- namely the rotation of points in a three dimensional coordinate system. In the case of the system of two dimensional rotation, the formal schemes associated with the treatment of the rotations of magnitudes (like Gauss' complex plane and numbers) are commutative precisely because the order of actions in such cases does not affect the result. However, in a three dimensional space, the order of actions (like rotations) does affect the final result. Thus, the formal system crafted to systematically describe such actions in 3 dimensional space must have rules/restrictions which reflect this. The fact that any mathematicians would be "shocked" by "numbers" which did not commute in this way, seems only to illustrate the problem of not distinguishing between collection logic and non-collection logic. The quaternions were created for the purpose of being physically useful, and, based on the physical properties of the thing they were supposed to apply to, their formal properties had to be modified to exclude commutation. What is so shocking about that?

Quaternions are not collections (numbers). They are formal devices used to furnish useful results about certain motions in three dimensions. They utilize collections, and thus can be said to be mathematical, but they are not themselves collections.

VIII. <u>Why Does Mathematics "Work"?</u>

Here arises a puzzle that has disturbed scientists of all periods: How is it possible that mathematics, a product of human thought that is independent of experience, fits so excellently the objects of physical reality? Can human reason, without experience, discover by pure thinking the properties of real things?

-Albert Einstein, Lecture: "Geometry and Experience", 1921

The above quote from Einstein respecting the question constituting the label of this chapter of our report serves the purpose of providing those unfamiliar with the problem a sense of how important and fundamental it is. The prominent American mathematician and teacher Morris Kline dedicated a chapter in his final book *Mathematics and the Search for Knowledge* to the question "Why does mathematics work?". In the course of providing an overview of the different ways in which prominent people have attempted to address that question, Kline points out that "Many mathematicians are happy to accept the remarkable applicability of mathematics but confess that they are unable to explain it.". In my view, a few philosophical distinctions must be made if we are to address the question in any meaningful way.

We must clarify what we mean by the word "work" in the question "Why does mathematics work?". There are a few different possible interpretations, each of which will offer different kinds of approaches determined by the epistemological standpoint one adopts. When we say that mathematics "works" we could mean: 1.) To indicate that mathematics is very conducive to the *description* of our phenomenal experiences. Or, 2.) To indicate that mathematics enables us to *predict* what phenomena we *will* experience. (This option presupposes the first one)

Let us first examine the question from the standpoint of interpretation 1: Why is mathematics very conducive to the *description* of our phenomenal experiences?

Based on this interpretation, one approach to the question could be that similar to the Aristotelian's, namely, that the mind undergoes experiences of phenomena, and from those phenomena -by virtue of repeated impressions of them- abstracts certain distinct characteristics. Once a certain kind of phenomenal characteristic is abstracted, it can be given a name and related to other phenomena so abstracted and named. If some of these abstracted characteristics are classified as "mathematical" then the question as to why mathematical concepts correspond to our phenomenal experiences is answered: Mathematical concepts are abstracted from experience (like all concepts) and thus, they correspond to our experiences. So much for that.

Another approach to answering this question would be the approach commonly called "Platonic": The mind contains within itself certain mathematical forms which are absolute. Thus, when we encounter phenomena, the phenomena is compared, by our minds, to these forms. Sometimes the phenomena are of such a characteristics as to provide us a very immediate access to the mathematical forms in our minds- a sort of reminder of something already in us. Thus, by this approach, one would say that mathematics is conducive to the description of our experiences because our experiences sometimes remind us of the absolute forms in our mind which are the stuff of mathematics. It could be said that based on how the fundamental principle of equality was discussed at the beginning of this report that my view on the matter is similar, in some aspects, to this "Platonic" one just described.

Another approach to the question interpreted in this way is that of Kant. If we admit that all phenomena which the human mind experiences are determined by certain laws and principles innate to the mind, then, if certain of those laws or principles are considered to be "mathematical" the question is thus answered: Mathematical concepts reflect the fundamental principles which are necessary to the mind's experience of any phenomena, and thus, phenomena will exhibit characteristics corresponding to mathematical concepts.

At the beginning of this report I provided a list of some of the fundamental operations and principles of the mind which are requisite to the very notion of conscious experience itself.

Accordingly, my own views can be said to generally correspond to the "Kantian" approach. Since the fundamental principles are the essential concepts of mathematics, the question as to why conscious experience corresponds to mathematical notions is thus addresses. I wish to say only a couple more things on this.

The process of perception is only a special case of the act of judgement. For the perception of anything is only possible if the mind judges something as a thing. It might be said that the instruments and standards of judgement are the fundamental principles of cognition, and that the mind can therefore form no judgements which do not rely on those principles. Therefore, no perceptions can take place which are not expressive of the fundamental principles- and thus "mathematical" in some way. However, the judgements which the mind makes about its experiences are subject to alteration, and therefore, the way things are perceived is subject to change. A change in perception is a change in the way the fundamental principles are manifested to our conscious awareness. Our perception of a distinct one thing might change to a perception of a many- but a many, in turn, of individual one things. Our perception of a magnitude of continuous quality might change to a perception of a magnitude heterogeneous- but a heterogeneity comprised of smaller magnitudes which, in turn, are themselves judged to be of some continuous characteristic over their extent.[64][65]

In the discussion of the fundamental principles near the beginning of the report, we provided illustrations as to how it is that the mind is unable to *conceive* of any experience without resorting to those principles. This influences the way we think about how judgment and perception must occur. But, another consideration here confronts us- consciousness and unconsciousness. Indeed, it seems to be the case that we are not able to make an absolute

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[64] Applying this kind of reasoning, respecting our phenomenal experience, to the way we conceive of the physical world, Leibniz pointed out that an infinite regression seems implied if we take reality to be composed of entities of discrete magnitude- an infinite regression which, unless we assume the existence of an "ultimate particle" (something he rejected) would seem to indicate that nothing really exists. This led Leibniz to resort to his conception of the Monad as the fundamental unit of physical reality, as opposed to an ultimate particle like those proposed by the atomists.

[65] The notion of the "phenomenal object" (as opposed to the physical object) is essentially a complex of memory associated with a particular perception which is recognized as similar to other perceptions experienced prior to it- perceptions always found to be associated with a certain pattern of changes in perception given changes in total phenomenal context- as that change in total phenomenal context is brought about by the willful actions of the perceiver, or not. For example, when someone says "that looks like a hard surface" they are not referring to some visual characteristic called "hardness", but, rather, to a notion, gathered from experience, that surfaces of certain visual characteristics are associated with the phenomenal sensation of "hardness" given certain phenomenal conditions (Like when our hand touches the surface). Or when someone claims that an element of their phenomenal experience is an "object", they are usually referring to the memories associated with that kind of perception- memories which lead them to think that the same potentials for changes in perception given the current perception are the same. For example, the ability of a visual object to be touched or rotated or moved given changes in the total phenomenal context- such as that person moving their body or another perceived "object" coming into "contact" with the object. From a purely phenomenal standpoint, an "object" is a certain kind of discrete perception which is associated with certain kinds of potentialities for change given different circumstances. The phenomenal conception of "object" is, of course, the basis for the usual notion of physical objects.

distinction between what we are conscious of, and what we are not conscious of. For example, we may be thinking about something very intently while things are happening around us which we pay absolutely no conscious attention to. Yet, if someone inquires of us if we remember perceiving some particular thing in our immediate environment while we were thus intently thinking, then it is very likely that we will be able to affirm that we experienced it, even though, at the time, we took absolutely no conscious note of it whatsoever. We may even be inclined to say that we *consciously perceived* it, and who could say that we did, or did not?. Besides this trivial hypothetical example which an experimental psychologist might find wanting, there are more rigorous experiments the results of which have placed the acceptance of the existence of unconscious perception on a firmer scientific basis. But, again, we cannot make an absolute distinction between unconscious and conscious perception- the boundary between them is too blurry.Thus, we have a problem. For, if we found that *conscious experience* is necessarily of the form required by the fundamental principles, what do we say of things which we "*unconsciously experience"*? Perhaps the implications of this problem, or lack thereof, will become clearer over time.

Now, returning to the original question which this chapter is intended to discuss, we will examine the second way in which that question can be interpreted: Why does mathematics enable us to *predict* what phenomena we *will* experience?

If we admit that mathematical notions are essentially constructs whose elements are the fundamental principles of cognition in various combinations with phenomenal notions/abstractions, then we seem to agree with the philosophy propounded by Kant and others, that the way in which the human mind is constructed necessitates the way in which that mind will be able to think about or experience anything. Thus, the reason mathematics corresponds to our experience is because our minds are "hardwired" to manifest consciousness in accordance with innate principles which are themselves the basis of mathematics. But, this does not explain the most incredible way in which mathematics "works"- namely, the *prediction* of phenomena *yet to be experienced*. If we admit that the fundamental principles of cognition are the basis of all consciousness/experience, as well as the substance of mathematical notions, then we only solve the problem as to why our *experience* corresponds to mathematical orderings and concepts. But, it does not explain why the use of mathematics can enable such accurate prediction of phenomena *yet to be experienced*.

In answering this question there might be readily identified three approaches: 1.) Someone who does not admit or denies the existence of a reality outside their experience (the phenomenalist)[66]. Someone of this epistemological category would necessarily take the position that the predictive

[66] Philosophically equivalent to this view would be someone who believes in a reality outside of experience, yet who does not believe in any causal connection between this reality and our experiences.

use of mathematics is, and will always be, a mystery. The other approach is as follows: 2.) Someone who hypothesizes a reality outside of their experience which exists and which is causally connected to the experiences we encounter (the epistemological dualist). This category has two subdivisions: A) Those who believe that the characteristics of this reality are not capable of being understood or known in any way (what we might call the "Kantian" approach); and, B) Those who believe that the characteristics of this reality are capable of being understood and known.

Let us first examine category 2A- the Kantians. The fact of the demonstrable progress of mathematical physics over the course of history puts the Kantian in a difficult position. For, if we admit that the world outside our experience is absolutely inconceivable and unknowable to our minds (as Kant did), but, yet, we find that the mathematical creations of our minds are able to accurately predict the occurrence of yet to be experienced phenomena -phenomena which we admit (as Kant did) to be *caused* by the action of physical reality on our minds- then we seem to almost directly imply, and loudly, that the mathematical notions we use for such prediction correspond, in some way, to aspects of the reality which lies outside of, and causes, our experiences. It seems quite difficult to attempt to maintain the philosophical position that every scientific and/or mathematical model/theory or device, used for the prediction of phenomena actually gives us *no* indication of any characteristic of reality whatsoever, and is, really, nothing more than a conceptual and/or formal *scheme* which *just so happens* to be utilizable for prediction of phenomena caused by the reality outside our experience.

Does the approach of the batch of philosophers adhering to the category of 2B render any result more satisfactory? The answer implied by this philosophical approach runs as follows: The reality which lies outside of our experience corresponds, in at least some of its characteristics, to our concepts, including our mathematical ones. Thus, mathematics enables us to predict the behavior of reality, and thus of our experiences which are immediately dependant upon the behavior of reality. Yet, ironically, though the progress of mathematical physics over history seems to suggest loudly a correspondence of our mathematical notions with reality in this way, the progress of modern physics also seems to suggest the opposite. The evolution of mathematical physics shows us a process of successive creation and destruction of theories about reality -which utilize mathematical and other notions- which are used for the purpose of predicting phenomena. These concepts of reality, these theories -which at one point in human history seemed to provide the most true way of thinking about the world and predicting its behavior- are *always* eventually supplanted by other concepts and mathematical models. People will often look back with amusement upon the old ideas about reality adhered to by past generations of scientists, thinking to themselves: "How could they possibly have believed the world is actually like that?". If anything can be said, it is that the mathematical notions we utilize in conceiving reality never really correspond to reality in a precise way. Einstein put it thus: "*As* *far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*" (This quote is the very next sentence after the quote presented as the introduction to this chapter found in the cited lecture).Therefore, it may be said that our mathematical notions of reality, strictly speaking, are never really *true*.

Thus, we seem to be caught in a paradoxical, or what might be better called *ironic*, position respecting the question as to whether our mathematical notions correspond to aspects of reality: On the one hand, the Kantian approach seems absurd, given the predictive power of mathematics. Yet, on the other hand, the history of mathematical physics impels us to concede that it is absurd to believe that any of our ideas about reality are actually, in the strictest terms, ever true.

The typical response by the 2B philosopher[67] to this uncomfortable ambiguity is to concede that we can never *precisely* know reality- but reality still exists and has definite objective characteristics which are, in principle, precisely knowable, the which can be known by us approximately and with indefinitely increasing degrees of precision. As long as the characteristics of reality are of close enough of an approximation to our basic mathematical notions as to be practically treated according to our mathematical idealizations, then the logic of our mathematical idealizations will be of practical and predictive use in the conception, description and prediction of reality.

However, even this view, so long and widely held by practicing scientists, has been losing adherents. This is due the modern development of the notion of the quantum and its implication of a theoretical limitation placed upon the precision with which it is possible for us to determine the characteristics of what we were wont to think of as reality; this, especially as it is considered in conjunction with the experimental findings (the double slit) which challenge any logical explanations consistent with the basic notions of reality which mankind has adhered to up to this point in his development.[68]

Furthermore, and more importantly, this view tacitly assumes that our conceptions respecting the *qualitative* features of reality are in *absolute* correspondence with reality. In other words: to say that the quantitative relations of things can never be known perfectly, but can be known indefinitely better, tacitly assumes that the "things" whose quantitative relations are to be thus known are actually characteristics of reality. But, as the history of science has shown us, it is not

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[67] The 2A philosopher has no problem with this paradox since they never tried to claim we know anything about reality anyway.
[68] For more of my thoughts on this, see my entry "Scientific Conceptualization and the Paradox of the Quantum" https://www.findingprometheus.com/single-post/2017/06/02/Scientific-Conceptualization-and-the-Paradox-of-the-Quantum

only our concepts of the quantitative relations of "things" in reality which are always subject to modification; *our concepts of the "things" themselves are always subject to modification*. Thus, the "things" we thought existed -the "things" whose quantitative relations we sought to approximate- turned out to not really have existed in the first place. Such is the guarantee of any "thing" conceived to "exist in reality".An expression of this reality of science was given by Einstein thus: "*The working theoretical physicist is not to be envied, because Mother Nature, or more precisely an experiment, is a resolute and seldom friendly referee of his work. She never says 'yes' to a theory, but only 'maybe' under the best of circumstances.*"

If, then, no definite idea we ever have about reality is really true, what is the reason that the successive adoption of new ideas about reality -as exemplified by the history of science- provide mankind with an increasing capacity to exist in reality? Is there anything about reality we can actually know? Is there any concept in our minds which can be said to share an identity with some aspect of reality "outside" our minds?

Lyndon LaRouche has provided an insight into this question:

"*Consequently, since formal scientific knowledge—existing deductive knowledge, is not permanent, it cannot contain within itself an adequate reflection of the lawful ordering of the universe.... That aspect of human knowledge which is proven to be in empirical correlation with man's increasing power over the universe through successive advances, is, only, the creative, preconscious processes through which revolutions in science and technology are successively, successfully attained....Only that aspect of the mind which is creative preconscious activity is, in terms of its continuing self development, in correspondence with the actual lawful ordering of the universe… The self-developing preconscious processes of mind, the creative work of the soul, are the only competent map of the universe we have accessible to us*"

The conceptions which mankind has had -or has- chosen to accept about reality, at any moment of his existence, may, by virtue of their usefulness to him over a given period, have been judged to truly correspond to reality. But, in reality, it were better to say, that those ideas never corresponded to reality, but, rather, they were ideas that The Creator intended us to consider were reality for a time, until new ideas, fostered by the conditions brought about by the elaboration of the old, were conceived; and so on, forever. The most truthful conscious notion, then, of the truth, of the reality of ourselves and of the universe, is the *power in us that created* the ideas which proved to be fruitful in their acceptance by mankind over some phase of his development. The most direct access to reality that we have is not the present state of theory about reality, but the power in us which generates, in us, the new idea which, when accepted and acted upon by mankind, proves to be *of the truth*.

Thus, those conceptions of reality which we were wont to claim true might yet be said to be truthful because they came from the truest reality we can know.